Reductions Between Problems in Reverse Mathematics and Computability Theory

Denis R. Hirschfeldt

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 $H \subseteq X$ is homogeneous for c there is an i < k s.t. every set in $[H]^n$ has color i.

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Patey: For $n \ge 2$, the complexity of solutions also increases with *k*.

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P and Q will denote such problems.

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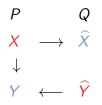
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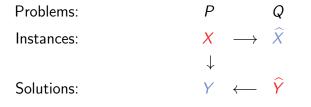
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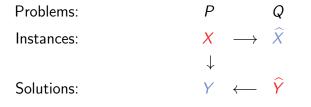
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Thm (Jockusch). $\mathsf{RT}_k^1 <_{\mathsf{c}} \mathsf{RT}_k^2 <_{\mathsf{c}} \mathsf{RT}_k^3 <_{\mathsf{c}} \mathsf{RT}_k^4 <_{\mathsf{c}} \cdots$

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Thm (Patey). $\operatorname{RT}_2^n <_{\operatorname{c}} \operatorname{RT}_3^n <_{\operatorname{c}} \operatorname{RT}_4^n <_{\operatorname{c}} \cdots$ for $n \ge 2$.

Weihrauch: P is Weihrauch reducible to Q, written $P \leq_w Q$, if there are Turing functionals Φ and Ψ s.t., for every instance X of P, Φ^X is an instance of Q, and for every solution \widehat{Y} to Φ^X , $\Psi^{X \oplus \widehat{Y}}$ is a solution to X. Weihrauch: P is Weihrauch reducible to Q, written $P \leq_w Q$, if there are Turing functionals Φ and Ψ s.t., for every instance X of P, Φ^X is an instance of Q, and for every solution \widehat{Y} to Φ^X , $\Psi^{X \oplus \widehat{Y}}$ is a solution to X.

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Thm (Rakotoniaina / Patey / Hirschfeldt and Jockusch). $RT_2^n <_w RT_3^n <_w RT_4^n <_w \cdots$.

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Write $P \equiv_{\omega} Q$ if $P \leqslant_{\omega} Q$ and $Q \leqslant_{\omega} P$.

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For a fixed *n*, we have $\mathsf{RT}_{j}^{n} \equiv_{\omega} \mathsf{RT}_{k}^{n}$ for all $j, k \ge 2$.

For $m, n \ge 3$, we have $\mathsf{RT}_k^m \equiv_{\omega} \mathsf{RT}_k^n$, but $\mathsf{RT}_k^1 <_{\omega} \mathsf{RT}_k^2 <_{\omega} \mathsf{RT}_k^3$. (Jockusch, Specker, Seetapun)

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If the game never ends then Player 1 wins.

If a player cannot make a move, the opponent wins.

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Let ID be $\forall X \exists Y Y = X$. Then $\mathsf{RT}^1_k \leq_{\mathsf{c}} \mathsf{ID}$ but $\mathsf{RT}^1_k \notin_{\mathsf{W}} \mathsf{ID}$.

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Thm (Hirschfeldt and Jockusch). $RT_k^1 \leq gW$ ID.

Theorem (Hirschfeldt and Jockusch). For $n \ge 3$ and $j \ge 1$, if

$$n+(j-1)(n-2) < m \leq n+j(n-2)$$

then

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 but $\mathsf{RT}^1_k \notin^m_{\mathsf{gW}} \mathsf{RT}^1_j$.

Patey determined the least m s.t. $\operatorname{RT}_{k}^{n} \leq_{\omega}^{m} \operatorname{RT}_{j}^{n}$ for $n \ge 2$ and j < k. For $n \ge 3$, this m is always 2. For n = 2 it goes to infinity with k. RT^n is $\forall k \operatorname{RT}_k^n$.

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The usual base theory RCA_0 consists of the basic axioms, Δ_1^0 -comprehension:

 $\forall n [\varphi(n) \leftrightarrow \psi(n)] \rightarrow \exists X \forall n [n \in X \leftrightarrow \varphi(n)]$ for all φ, ψ s.t. φ is Σ_1^0 and ψ is Π_1^0 , and X is not free in φ , and Σ_1^0 -induction:

$$(\varphi(0) \land \forall n [\varphi(n) \rightarrow \varphi(n+1)]) \rightarrow \forall n \varphi(n)$$

for all Σ_1^0 formulas φ .

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 $\mathsf{RT} \leqslant_{\omega} \mathsf{RT}_2^3$ but $\mathsf{RCA}_0 \nvDash \mathsf{RT}_2^3 \to \mathsf{RT}$.

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The notions of instance and solution of a problem still make sense over any structure \mathcal{N} in the language of first-order arithmetic.

For $X_0, \ldots, X_n \subseteq |\mathcal{N}|$, let $\mathcal{N}[X_0, \ldots, X_n] = (\mathcal{N}, S)$ where S consists of all subsets of $|\mathcal{N}|$ that are Δ_1^0 -definable from parameters in $|\mathcal{N}| \cup \{X_0, \ldots, X_n\}$.

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Player 1: A model (\mathcal{N}, S) of RCA₀ with $|\mathcal{N}|$ countable, and a *P*-instance $X_0 \in S$.

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Player 2: Either a solution to X_1 in $\mathcal{N}[X_0, X_1, X_2]$, or a Q-instance $Y_3 \in \mathcal{N}[X_0, X_1, X_2]$.

Thm (Hirschfeldt and Jockusch / Dzhafarov, Hirschfeldt, and Reitzes). If $RCA_0 \vdash Q \rightarrow P$ then Player 2 has a winning strategy for $G^{RCA_0}(Q \rightarrow P)$. Otherwise, Player 1 does.

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Thm (Dzhafarov, Hirschfeldt, and Reitzes). If $RCA_0 \vdash Q \rightarrow P$ then there is an $n \in \omega$ s.t. Player 2 has a winning strategy for $G^{RCA_0}(Q \rightarrow P)$ that wins in at most *n* many moves.

The least such n can be seen as measuring the minimal number of applications of Q needed in proving P over RCA₀.

P is generalized Weihrauch reducible to *Q* over *RCA*₀, written $P \leq_{gW}^{RCA_0} Q$, if Player 2 has a computable (i.e., Δ_1^0) winning strategy for $G^{RCA_0}(Q \to P)$.

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Theorem (Dzhafarov, Hirschfeldt, and Reitzes). If $P \leq_{gW}^{RCA_0} Q$ then there is an $n \in \omega$ s.t. Player 2 has a computable winning strategy for $G^{RCA_0}(Q \to P)$ that wins in at most *n* many moves.

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We can also define computable reducibility and Weihrauch reducibility over RCA₀ using 2-move games.

Reductions Between Problems in Reverse Mathematics and Computability Theory

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