Reductions Between Problems in Reverse Mathematics and Computability Theory

Denis R. Hirschfeldt

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A *k-coloring* of $[X]^n$ is a map $c : [X]^n \to \{0, 1, \ldots, k-1\}$.

$H \subseteq X$ is *homogeneous* for c there is an $i < k$ s.t. every set in $[H]^n$ has color i.

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Patey: For $n \geqslant 2$, the complexity of solutions also increases with k.

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P and Q will denote such problems.

Dzhafarov: P is *computably reducible* to Q, written $P \leqslant_{\mathrm{c}} Q$, if

for every instance X of P,

there is an X-computable instance $\widehat{X}$ of Q s.t.,

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Thm (Rakotoniaina / Patey / Hirschfeldt and Jockusch).
$\mathrm{RT}^n_2 <_{\mathrm{W}} \mathrm{RT}^n_3 <_{\mathrm{W}} \mathrm{RT}^n_4 <_{\mathrm{W}} \cdots$.

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For $m, n \geqslant 3$, we have $\mathsf{RT}^m_k \equiv_\omega \mathsf{RT}^n_k$, but $\mathsf{RT}^1_k <_\omega \mathsf{RT}^2_k <_\omega \mathsf{RT}^3_k$.
(**Jockusch, Specker, Seetapun**)

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If the game never ends then Player 1 wins.

If a player cannot make a move, the opponent wins.

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Third Move:

Player 1: A solution X_2 to Y_2.

Player 2: Either an $(X_0 \oplus X_1 \oplus X_2)$-computable solution to X_0, or an $(X_0 \oplus X_1 \oplus X_2)$-computable Q-instance Y_3.

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Patey determined the least m s.t. $\mathrm{RT}^n_k \leqslant_{\omega}^{m} \mathrm{RT}^n_j$ for $n \geqslant 2$ and $j < k$.

For $n \geqslant 3$, this m is always 2. For $n = 2$ it goes to infinity with k.

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$\mathrm{RT}^1 \leqslant_{\mathrm{gW}} \mathrm{RT}^1_2$ but $\mathrm{RT}^1 \not\leqslant^j_{\mathrm{gW}} \mathrm{RT}^1_2$ for all j.

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The usual base theory RCA_0 consists of the basic axioms,

Δ^0_1-*comprehension*:

$$\forall n\,[\varphi(n) \leftrightarrow \psi(n)] \rightarrow \exists X\,\forall n\,[n \in X \leftrightarrow \varphi(n)]$$

for all φ, ψ s.t. φ is Σ^0_1 and ψ is Π^0_1, and X is not free in φ,

and Σ^0_1-*induction*:

$$(\varphi(0) \wedge \forall n\,[\varphi(n) \rightarrow \varphi(n+1)]) \rightarrow \forall n\,\varphi(n)$$

for all Σ^0_1 formulas φ.

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$\mathsf{RT} \leqslant_\omega \mathsf{RT}^3_2$ but $\mathsf{RCA}_0 \nvdash \mathsf{RT}^3_2 \to \mathsf{RT}$.

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The notions of instance and solution of a problem still make sense over any structure $\mathcal{N}$ in the language of first-order arithmetic.

For $X_0, \dots, X_n \subseteq |\mathcal{N}|$, let $\mathcal{N}[X_0, \dots, X_n] = (\mathcal{N}, S)$ where S consists of all subsets of $|\mathcal{N}|$ that are Δ^0_1-definable from parameters in $|\mathcal{N}| \cup \{X_0, \dots, X_n\}$.

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Third Move:

Player 1: A solution X_2 to Y_2 in S.

Player 2: Either a solution to X_1 in $\mathcal{N}[X_0, X_1, X_2]$, or a Q-instance $Y_3 \in \mathcal{N}[X_0, X_1, X_2]$.

$\vdots$

Thm (Hirschfeldt and Jockusch / Dzhafarov, Hirschfeldt, and Reitzes). If $\mathsf{RCA}_0 \vdash Q \to P$ then Player 2 has a winning strategy for $G^{\mathsf{RCA}_0}(Q \to P)$. Otherwise, Player 1 does.

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Thm (Dzhafarov, Hirschfeldt, and Reitzes). If $\mathsf{RCA}_0 \vdash Q \to P$ then there is an $n \in \omega$ s.t. Player 2 has a winning strategy for $G^{\mathsf{RCA}_0}(Q \to P)$ that wins in at most n many moves.

The least such n can be seen as measuring the minimal number of applications of Q needed in proving P over RCA_0.

P is *generalized Weihrauch reducible* to Q *over RCA_0*, written $P \leqslant_{\mathrm{gW}}^{\mathrm{RCA}_0} Q$, if Player 2 has a computable (i.e., Δ^0_1) winning strategy for $G^{\mathrm{RCA}_0}(Q \to P)$.

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Let $j, k \geqslant 2$. Then $\mathsf{RT}^1 \equiv_{\mathrm{gW}} \mathsf{RT}^1_j \equiv_{\mathrm{gW}} \mathsf{RT}^1_k$, and $\mathsf{RT}^1_j \equiv_{\mathrm{gW}}^{\mathsf{RCA}_0} \mathsf{RT}^1_k$, but Hirst showed that $\mathsf{RCA}_0 \nvdash \mathsf{RT}^1$, so $\mathsf{RT}^1 \nleqslant_{\mathrm{gW}}^{\mathsf{RCA}_0} \mathsf{RT}^1_k$.

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Theorem (Dzhafarov, Hirschfeldt, and Reitzes). If $P \leqslant_{\mathrm{gW}}^{\mathrm{RCA}_0} Q$ then there is an $n \in \omega$ s.t. Player 2 has a computable winning strategy for $G^{\mathrm{RCA}_0}(Q \to P)$ that wins in at most n many moves.

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We can also define computable reducibility and Weihrauch reducibility over RCA_0 using 2-move games.

Reductions Between Problems in Reverse Mathematics and Computability Theory

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