On the reverse philosophy of the sorites paradox

University of Connecticut
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Walter Dean

University of Warwick &
Institut d'études avancées de Paris

Outline

Part One: On "reverse philosophy".

- What? The application of ***reverse mathematics*** to arguments in contemporary analytic philosophy.
- Why? The Carrot and The Stick.

Part Two: The sorites as an example of reverse philosophy.

- Role of mathematical representation / ***measurement theory***.
- Different forms of the sorites:
  - Conditional for $heap$: $\mathsf{I}\Delta_0 + \mathsf{Exp}$
  - Conditional for $tall$: RCA_0
  - Continuous for red: ACA_0
  - Topological/vicinity-based for $game$: ACA_0 (at least)
- Morals and responses illustrating the utility of the methodology.

The basic phenomenon and reverse mathematics

- The "basic phenomenon": existence of ***philosophical arguments*** which contain ***mathematical theorems*** amongst their premises.
- Canonical example: Kreisel's squeezing argument.*

P_1: If $\Gamma \vdash \varphi$, then the argument $\Gamma \therefore \varphi$ is $\overbrace{\textit{intuitively valid}}^{\text{analysandum}}$.
P_2: If the argument $\Gamma \therefore \varphi$ is intuitively valid, then $\Gamma \models \varphi$.
P_3: If $\Gamma \models \varphi$, then $\Gamma \vdash \varphi$.
C: The argument $\Gamma \therefore \varphi$ is intuitively valid if and only if $\Gamma \models \varphi$.

- P_3 is the Gödel Completeness Theorem (GCT) for FOL.
  - Expressible in the ***language of 2nd-order arithmetic*** $\mathcal{L}_2$.
  - $\mathrm{RCA}_0 \vdash \mathrm{GCT} \leftrightarrow \mathrm{WKL}$
    - WKL: "every infinite subtree of $2^{\mathbb{N}}$ has an infinite path".
    - This implies the existence of non-computable sets.
  - Interested parties: Kreisel, Quine, Dummett, Troelstra, van Dalen, Etchemendy, Boolos, Field, Halbach, Beall (?) ...

*Cf. also Kreisel 1960 on $\mathrm{HYP} = \textit{predicative} = \Delta^1_1$ via Kleene's Theorem.

Why care?

- The basic phenomenon is ***pervasive*** – e.g.
  - Philosophy of mathematics: Löwenheim-Skolem theorem and Dedekind/Zermelo categoricity theorems ...
  - Philosophy of logic: Theories of truth, fixed point theorems, ...
  - Philosophy of science: Beth's theorem, interpretability, ...
  - Metaphysics: Modal logic, mereology, ...
  - Formal epistemology: Dutch book theorems, ...
  - Political philosophy/ethics : Impossibility theorems (e.g. Arrow, Gibbard-Satterthwaite), Harsanyi's Util. Theorem, ...
  - Philosophy of language/semantics: Representation theorems (e.g. Hölder, Debreu), Montague grammar/types, ...
- No systematic accounting has been made. (?)
- Reverse math is well-suited to the relevant "ordinary" maths.
- To philosophy: **methodological benefits / costs** ...

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- To reverse mathematics: ***New sources of reversals.***

The Carrot and The Stick

- The Carrot:
  - Discover premises equivalent to principles whose strength we can analyze via the methods of ***Reverse Mathematics***.
  - So arguments have *more structure* than realized – e.g. ***novel rhetorical options***, ***idealizations made explicit***.
  - A revitalization of the ***Hilbert Program*** within philosophy.
    - ***Arithmetic/computability theory*** rather than *set theory*.
    - Solutions to philosophical problems within (***classical***) ***mathematics*** rather than (*non-classical*) *logic*.
- The Stick:
  - A ***proponent*** of an argument is someone who wishes to use it to infer its conclusion from its premise.
    - Such a theorist is thus committed to its ***soundness***.
    - If $\mathrm{B} \vdash \mathrm{P}_i \leftrightarrow \Phi_i$, they are also committed to the ***truth*** of Φ_i.
  - A novel ***philosophical indispensability argument***.
    - The proponents of certain "fancy" arguments in contemporary analytic philosophy have mathematical commitments.
    - So there is a tension between "fancy analytic philosophy" and nominalism/fictionalism.

Reverse philosophical analysis

Step 1 : Take an argument $\Gamma \therefore \varphi$ "from the literature".

Step 2 : Regiment the argument in *standard form*.

$$\begin{array}{lll} \mathrm{P}_1 & & \text{(Philosophical premise)} \\ \vdots & & \\ \mathrm{P}_i & \mapsto \Pi_i \Leftrightarrow \Phi_i & \text{(Mathematical/mixed principle)} \\ \vdots & & \\ \mathrm{P}_\mathrm{n} & & \\ \hline \mathrm{C} & & \text{(Philosophical conclusion)} \end{array}$$

Step 3 : Formalize the relevant mathematical and mixed premises P_i as statements Π_i in $\mathcal{L}$.

Step 4 : Determine if Π_i ***reverses*** to a recognized mathematical principle Φ_i over a ***base theory*** B – i.e. $\mathsf{B} \vdash \Pi_i \leftrightarrow \Phi_i$?

Step 5 : Conclude (via an indispensability-like argument) that accepting Φ_i is thereby a condition for accepting the argument.

Step 6 : Assess the philosophical/methodological consequences.

All you need to know about Reverse Math (for this talk)

- A *generic* reverse mathematical program consists in:
  - An identification of a language $\mathcal{L}$ and a base theory B.
  - Some "ordinary theorems" Ψ and "axioms" Φ s.t. B $\vdash \Psi \leftrightarrow \Phi$.
  - E.g. ZF $\vdash \mathrm{WO} \leftrightarrow \mathrm{AC}$.
- The *specific* Reverse Mathematical program of SoSOA:
  - $\mathcal{L}$ is $\mathcal{L}_2 = \{0, 1, +, \times, <\}$ with 2nd-order variables, quantifiers
  - B is $\mathsf{RCA}_0 = \mathsf{Q} + \mathrm{Ind}(\Sigma^0_1) + \Delta^0_1\text{-CA}$
    - $\Delta^0_1\text{-CA}$ is the comprehension scheme limited to Δ^0_1-formulas.
    - The minimal ω-model of RCA_0 is $\mathcal{R} = \langle \mathbb{N}, \textsc{Rec}, 0, 1, +, \times, < \rangle$ with $\textsc{Rec}$ = the ***computable*** sets $X \subseteq \mathbb{N}$.
  - $\mathsf{ACA}_0 = \mathsf{RCA}_0 + \mathrm{ACA}$
    - ACA is the comprehension scheme limited to *arithmetical formulas* (i.e. 1st-order with 2nd-order parameters).
    - The minimal ω-model of RCA_0 is $\mathcal{A} = \langle \mathbb{N}, \textsc{Arith}, 0, 1, +, \times, < \rangle$ with $\textsc{Arith}$ = the ***arithmetically definable*** $X \subseteq \mathbb{N}$.
  - $\mathsf{RCA}^*_0 \subsetneq \mathsf{RCA}_0 \subsetneq \mathsf{WKL}_0 \subsetneq \mathsf{ACA}_0 \subsetneq \mathsf{ATR}_0 \subsetneq \Pi^1_1\text{-CA}_0 \subsetneq \ldots$
    - Is this sequence *canonical*? What does it *track*?
    - See, e.g., D. & Walsh 2017, Eastaugh 2019.

Arguments and reversals

Argument	Principle/Theorem	Theory
Finite democracy	Finite Arrow's Theorem	$\mathsf{I}\Delta_0 + \exp = \mathsf{EFA}$
"Justification of deduction"	Soundness of FOL	$\mathsf{I}\Delta_0 + \mathsf{supexp}$
Conditional sorites	Hölder's Theorem$^{\pm}$	RCA_0
Dutch book	Hyperplane Separation	RCA_0 or WKL_0
Squeezing validity	Completeness of FOL	WKL_0
Category of $\mathbb{N}$	Dedekind's Theorem	WKL_0
Nominalization	Arithmetized Completeness	$\mathsf{Con}(\mathsf{N}^*) + \mathsf{WKL}_0$
Infinite democracy	Fishburn/K&S Thms	ACA_0
Continuous sorites	Sup/Inf Principles	ACA_0
Squeezing predicativity	Kleene's Theorem	ACA_0
Metaphysical Universality	Exist canonical model	ACA_0
Untyped truth	Exist least fixed point	$\Pi^1_1\text{-}\mathsf{CA}_0$
Field's theory of truth		$\Pi^1_3\text{-}\mathsf{CA}_0$

Forms of the sorites paradox

Form	Ex.	Theorem/Principle	System	Reference
QF-conditional	*heap*	Cut (?)	Propositional logic	e.g. Sazonov 1995
∀-cond. collective	*heap*	$\forall x \exists y (2^x = y)$	$\mathrm{I}\Delta_0 + \mathrm{Exp}$	D. 2018
Inductive	*heap*	$\mathrm{Ind}(\mathcal{L}_P)$	$\mathrm{I}\Delta_0/\mathcal{L}_P + \mathrm{Exp}$	D. 2018
Line drawing	*heap*	$\mathrm{LNP}(\mathcal{L}_P)$	$\mathrm{I}\Delta_0/\mathcal{L}_P + \mathrm{Exp}$	D. 2018
∀-cond. graded	*tall*	Hölder's Theorem$^{\pm}$	RCA_0	Solomon 1998/9
Continuous	*green*	Sup + Debreu	ACA_0	Weber&Colyvan 2011/21
Continuous	*tall*	Sup + Hölder	ACA_0	Weber&Colyvan 2011/21
Topological	*game*	Conn+LC ⇒ GC	ACA_0 (at least)	Weber&Colyvan 2011/21
Vicinity-based	???	3.5 of D&D 2010	ACA_0	Dzhafarov 2019

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Please keep in mind:

- I don't really care about the sorites / vagueness / etc.
  - (If you don't like this example, we've got many others ...)
- My goal is convincing "*paradox mongers*" (you?) that regarding the sorites as a *classically valid argument* has mathematical commitments.
  - (Which they are welcome to *deny* ...)
- This requires attending to details of the ***linguistic formulations*** of the various forms.

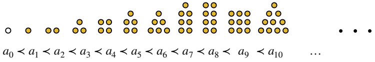
Universal conditional sorites for collective nouns

$P_1 : \neg \mathrm{Heap}(a_0)$

$P_2 : \forall i(\neg \mathrm{Heap}(a_i) \rightarrow \neg \mathrm{Heap}(a_{i+1}))$

$C : \neg \mathrm{Heap}(a_{10000})$

$\neg \mathrm{Heap}(a_0)$



$\neg \mathrm{Heap}(a_{10000})$

a_{10000}

A

Universal conditional sorites for collective nouns

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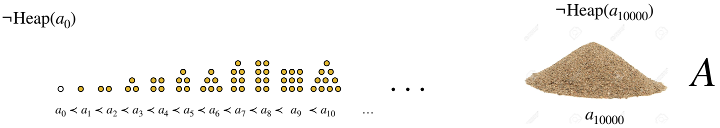

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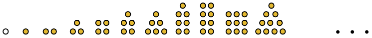
$\Pi_1 : \neg \text{Heap}^*(0)$

$\Pi_2 : \forall x(\neg \text{Heap}^*(x) \rightarrow \neg \text{Heap}^*(x+1))$

$K : \neg \text{Heap}^*(\overline{10000})$

$\text{Heap}^*(n)$ iff a_n is composed of n units (*grains*)

$\neg \text{Heap}(a_0)$



$a_0 \prec a_1 \prec a_2 \prec a_3 \prec a_4 \prec a_5 \prec a_6 \prec a_7 \prec a_8 \prec a_9 \prec a_{10}$ ...

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$\Pi_1 : \neg \mathrm{Heap}(a_0)$

$\Pi_2 : \forall x \forall y((\neg \mathrm{Heap}(x) \wedge f(x) = y) \rightarrow \neg \mathrm{Heap}(f^{-1}(y+1)))$

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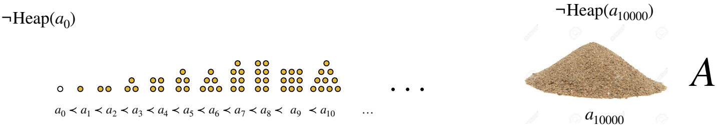
For all $a \in A$, define $f(a) = n$ iff a is composed of n units.



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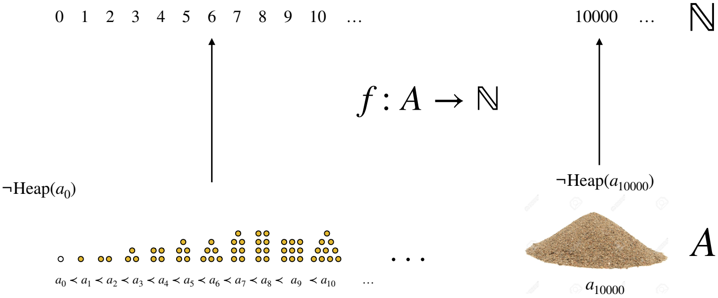

Mathematical representation

Basic claim: Felicitously formalizing even simple forms of the sorites requires a ***representation theorem***.

Proposition 1: Suppose that $\mathcal{A} = \langle A, \prec \rangle$ is a finite discrete linear order. Then there is a unique $n \in \mathbb{N}$ and bijection $f : A \rightarrow \{0, ..., n-1\}$ s.t.

i) $a_i \prec a_j$ iff $f(a_i) < f(a_j)$

ii) a_j is the immediate $\prec$-successor of a_i iff $f(a_j) = f(a_i) + 1$

We can now "officially" formulate the following argument:

Π_1: $\neg\mathrm{Heap}(a_0)$

Π_2: Proposition 1

Π_3: $\forall x \forall y((\neg\mathrm{Heap}(x) \wedge \rightarrow f(x) = y) \rightarrow \neg\mathrm{Heap}(f^{-1}(y+1)))$

Π_4: Addition axioms of Q

K : $\neg\mathrm{Heap}(a_{10000})$

The first level of mathematical involvement

- Other examples of mathematical representation in arguments:
  - Gödel numbering, modality via Kripke semantics, preference as relations or utilities, credences as probability measures, ...
- Arguments illustrating the ***1st level of mathematical involvement***:
  - Can't be formulated/applied w/o mathematical representation.
  - Typically involves expanding signature/schema to include *mathematical* and *mixed expressions* – e.g. $0, 0', \ldots, f(x), f^{-1}(x)$.
  - Axioms governing them must then be added – e.g. Q.
  - But the arguments are ***enthymemic*** w/o representation thms.
  - So where are they provable?
- Fact: Proposition 1 is provable in RCA_0.
  - If $A = \langle a_0, \ldots, a_{n-1} \rangle$ we can defn $s(a_i) = a_{i+1}$ by b'd min.
  - Then $f(a_0), f(s(a_i)) = i + 1$ exists by $\Delta^0_1\text{-CA}$ and $\text{Ind}(\Sigma^0_1)$.
- Upshots:
  - Assumes a ***structuralist*** understanding of representation thms.
  - But can now take RCA_0 as a formal premise in the argument.

From collective nouns to magnitude-related adjectives

Terminology from philosophy of language / linguistics:

1) *Collective nouns*: *heap*, *bald*, *forest*...
2) *Gradable* (or *scalar*) adjectives:
   a) Magnitude-related: *tall* (length), *heavy* (mass), *brief* (time), ...
   b) Non-magnitude-related: *happy*, *nice*, *important*, ...

From collective nouns to magnitude-related adjectives

Intuitively, a hundredth of an inch cannot make a difference to whether or not a man counts as tall – such tiny variations, undetectable using the naked eye and everyday measuring instruments ... So we have the principle

(Tol_{tall}) If x is tall, and y is only a hundredth of an inch shorter than x, then y is also tall.

But imagine a line of men, starting with someone seven feet tall, and each of the rest a hundredth of an inch shorter than the man in front of him. Repeated applications of (Tol_{tall}) ... imply that each man we encounter is tall, however far we continue. And this yields a conclusion which is clearly false, namely that a man less than five feet tall, reached after three thousand steps ... is also tall.

Keefe 2000

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Hallmarks of ***extensive measurement*** ...

Extensive measurement

Suppose we have an empirical structure $\mathcal{A} = \langle A, \preceq, \circ \rangle$ which we would like to reason about via $f : A \rightarrow \mathbb{R}^+$ s.t.

i) $a \preceq b$ iff $f(x) \leq f(b)$

ii) $f(a \circ b) = f(a) + f(b)$

What are necessary and sufficient conditions on $\mathcal{A}$ for $f(x)$ to exist?

Hölder's Theorem$^{\pm}$

Let $\mathcal{A} = \langle A, \preceq, \circ \rangle$ be s.t. $A \neq \emptyset$, $\preceq$ on $A^2, \circ : A^2 \rightarrow A$. Then there is $f(x)$ *representing* $\mathcal{A}$ *in* $\mathcal{R} = \langle \mathbb{R}^+, \leq, + \rangle$ – i.e. i) and ii) hold – iff

i) $\preceq$ is a weak order – i.e. reflexive, transitive, and connected;

ii) Weak associativity: $a \circ (b \circ c) \sim (a \circ b) \circ c$ for all $a, b, c \in A$;

iii) Monotonicity: $a \preceq b$ iff $a \circ c \preceq b \circ c$ iff $c \circ a \preceq c \circ b$ for all $a, b, c \in A$;

iv) Archimedean: For all $a, b \in A$, if $a \preceq b$, then for all $c, d \in A$, there exists $n \in \mathbb{N}$ such that $\overline{n}a \circ c \preceq \overline{n}b \circ d$.

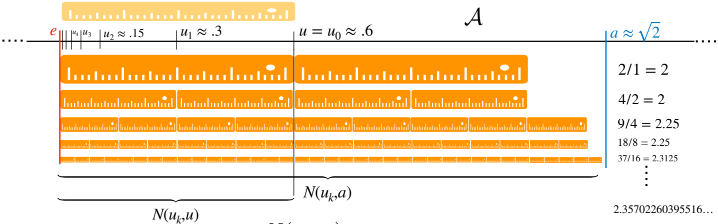
Moreover, if $f' : A \rightarrow \mathbb{R}^+$ also satisfies i), ii), then $\exists c > 0$ s.t. $f'(x) = cf(x)$.

Hölder's Theorem$^{\pm}$ in RCA$_0$

- Hölder's original theorem (Hölder 1901):
  - Stronger assumptions: $\mathcal{A}$ is an ordered Archimedean group satisfying the Dedekind property.
  - Stronger conclusion: $f(x)$ is ***onto*** to $\mathbb{R}$.
- Hölder's Theorem$^{\pm}$ (Krantz 1968 & et al. 1971, Roberts & Luce 1968)
  - If we don't require *onto*, weaker assumptions about $\mathcal{A}$ suffice.
  - In this case the proof is ***constructive***.
- Hölder's Theorem^{-}: If $\mathcal{A}$ is an ordered Archimedean group, then $\mathcal{A}$ is order isomorphic to a subgroup of $\langle \mathbb{R}^{+}, \leq, + \rangle$.
- Solomon (1998) showed Hölder's Thoerem^{-} is provable in RCA$_0$.
  - Archimedean: For all $a, b \in A$, there exists $n \in \mathbb{N}$ s.t. $\overline{n}a =_{\text{df}} \overbrace{a \circ \ldots \circ a}^{n \text{ times}} \succeq b$.
  - The proof can be adapted to show that Krantz et al. 1971's proof of Hölder's Theorem$^{\pm}$ can also be carried in RCA$_0$.

On the proof of Hölder's Thoerem$^-$

- For $u \prec a$ let $N(u,a) =$ the unique $n \in \mathbb{Z}$ s.t. $\overline{n}u \preceq a \prec \overline{(n+1)}u$.
- Suppose $A^+ = \{a \in A : a \succ e\}$ does not contain a $\prec$-least elt.
  - For any "unit" $u \in A^+$, there is $\langle u_k : k \in \mathbb{N}\rangle$ s.t. $\overline{2}u_{k+1} \preceq u_k$.
  - As if $y \prec u_k$, then y or $u_k \circ y^{-1}$ will be $\preceq$ "half" of x_k.
- Fixing $u \prec a$, we can now "measure" $a \in A^+$ as follows:



- Define $f(a) = \lim_{k\to\infty} \frac{N(u_k,a)}{N(u_k,u)} \in \mathbb{R}$ and observe $N(u_k,u) \geq 2^k$.
  - Thus if $q_k = \frac{N(u_k,a)}{N(u_k,u)}$, then $|q_{k+1} - q_k| \leq 2^{-k}$.
- $\langle q_k \mid k \in \mathbb{N}\rangle$ is a constructive, fast-converging Cauchy seq.
  - This is ***exactly*** how a real number r is defined in RCA_0.

The mathematical involvement of the sorites for *tall*

- Claim: It is not possible to ***convincingly formulate*** a sorites argument for $tall$ (e.g.) without measurement theoretic vocabulary.
- E.g. this is what allows us to formulate Tolerance as

$$(\text{Tol}^*_{tall}) \quad \forall x \in A \forall y \in A(\text{Tall}(x) \wedge |f(x) - f(y)| < 0.01 \rightarrow \text{Tall}(y))$$

- So (something like) Hölder's Theorem$^{\pm}$ becomes a formal premise.
- But this requires both empirical and mathematical axioms.
  - Positivity: $a \prec a \circ b \;\; \Rightarrow$ empirical domain is infinite. (Required?)
  - But since $\text{RCA}_0 \vdash$ Hölder$^{\pm}$, all $r \in \text{ran}(f)$ can be ***computably approximated*** as $\langle q_k : k \in \mathbb{N} \rangle$.
- So we do not broach the ***2nd level of math. involvement***.
  - I.e. the argument does ***not*** assume properties of $\mathbb{R}$ beyond those *given in its* (standard Reverse Mathematical) *definition*.

From the conditional sorites to the continuous sorites

- Common intuition: Discretization into units – e.g. .01 inch – does not do justice to our intuitions about how vague gradable adjectives are "***insensitive to small changes***".
- Paradigmatically true of "***perceptual continua***":

Imagine a patch darkening **continuously** from white to black. At each moment during the process the patch is darker than it was at any earlier moment. Darkness comes in degrees. The patch is dark to a greater degree than it was a second before, even if the difference is **too small to be discriminable by the naked eye**. Given that there are as many moments in the interval of time as there are real numbers between 0 and 1, **there are at least as many degrees of darkness as there are real numbers between 0 and 1, an uncountable infinity of them**. Such numbers can be used to **measure degrees of darkness**.

Williamson 1994, p. 113

- Same presumably also makes sense for magnitude-related adjectives like *tall* – e.g. when restricted to points on a ruler.
  - (Important because it's less clear what representation theorem is appropriate for color, intensity, etc.)

How to generalize tolerance?

- Three mathematically-inspired options:
  1) Metrical: Williamson 1994, Weber & Colyvan 2010, Weber 2021
  2) Topological: Weber & Colyvan 2010
  3) Vicinity-based: Dzhafarov & Dzhafarov 2010a/b

How to generalize tolerance?

- Three mathematically-inspired options:
  1) Metrical: Williamson 1994, Weber & Colyvan 2010, Weber 2021
  2) Topological: Weber & Colyvan 2010
  3) Vicinity-based: Dzhafarov & Dzhafarov 2010a/b
- Why 1) over 2) and 3)?
  - Options 2) and 3) collapse the distinction between changes over "one step" and over "many steps". (Rizza 2013)
    - So it's unclear if the results are "paradoxes of vagueness" or just (necessary) mathematical facts about mappings from the relevant spaces to "degrees".
  - But the proofs of the relevant "anti-representation theorems" are still mathematically involved to (at least) the same extent. (Dhzafarov 2019)

The Leibniz Continuity Condition

Leibniz (1687): Nature does not make jumps. *(Natura non saltum facit.)*

L'Huilier (1787): If a variable quantity at all stages enjoys a certain property, its limit will enjoy the same property.

Priest (2006): Given any limiting process, whatever holds up to the limit holds at the limit.

Recall:

- $\mathcal{X} \subseteq \mathbb{R}$ is *bounded above* (BA) if $\exists r \forall s \in \mathcal{X}(s \leq r)$.
- Presuming $\mathcal{X}$ is bounded above, we define

$$\sup(\mathcal{X}) = \textit{the least upper bound of } \mathcal{X}.$$

Weber & Colyvan 2010:

- Suppose that $\varphi(x)$ is a vague predicate with field C and that $f : C \to [s,t]$ – a closed interval of $\mathbb{R}$ – is a ***bijection***.

(LCC) $\forall \mathcal{X} \subseteq \mathbb{R}(\text{BA}(\mathcal{X}) \wedge \forall r(r \in \mathcal{X} \to \varphi(f^{-1}(r))) \to \varphi(f^{-1}(\sup(\mathcal{X}))))$

- This is ***not*** a valid schema – e.g. take $\varphi(f^{-1}(r))$ iff r *is rational*.

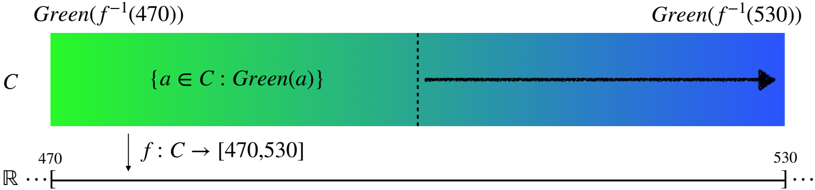
The continuous sorites as an argument

- "Set up" axioms Σ:
  - $P(x)$ a vague predicate – e.g. *green*.
  - C a "continuum" for $P(x)$ – e.g. green-blue spectrum
  - $\prec$ a total order on C – e.g. *less green than* – s.t.
    $$P(a) \wedge b \prec a \rightarrow P(b)$$
  - $f : C \rightarrow [s,t]$ a bijection and $a \prec b \rightarrow f(a) < f(b)$.
- W&C's formulation as an argument in standard form:

  P_1: $P(f^{-1}(s))$

  P_2: $\forall \mathcal{X} \subseteq \mathbb{R}(\mathrm{BA}(\mathcal{X}) \wedge \forall r(r \in \mathcal{X} \rightarrow \varphi(f^{-1}(r))) \rightarrow \varphi(f^{-1}(\sup(\mathcal{X}))))$

  C: $P(f^{-1}(t))$
- What is this supposed to show?



Argument or *proof*?

P_1: $P(f^{-1}(s))$

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P_3: $\forall \mathcal{X} \subseteq \mathbb{R}(\mathrm{BB}(\mathcal{X}) \wedge \forall r(r \in \mathcal{X} \to \varphi(f^{-1}(r))) \to \varphi(f^{-1}(\inf(\mathcal{X}))))$

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How do we get from $\mathrm{P}_1, \mathrm{P}_2, \mathrm{P}_3$ to C?

Argument or *proof*?

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How do we get from $\mathrm{P}_1, \mathrm{P}_2, \mathrm{P}_3$ to C? (Reconstructing Weber 2021 based on Chase)

1) Suppose Σ, $P(f^{-1}(s))$ and for a contradiction $\neg P(f^{-1}(t))$.
2) Define the following subsets of $[s, t] \subseteq \mathbb{R}$:

$$U = \{f(x) : P(x) \ \wedge \ x \in C\} \qquad V = \{f(x) : \neg P(x) \ \wedge \ x \in C\}$$

3) $U, V \neq \emptyset, [s, t] = U \sqcup V$, U bounded above, V bounded below.
4) By the Supremum and Infimum Principles, $\sup(U), \inf(V)$ exist.
5) So $P(f^{-1}(\sup(U)))$ by P_2 and $\neg P(f^{-1}(\inf(X)))$ by P_3.
6) Since $<$ is a linear order on $\mathbb{R}$, we must have one of i) $\sup(U) < \inf(V)$ or ii) $\inf(V) < \sup(U)$ or iii) $\sup(U) = \inf(V)$.

i-ii) $\exists r \in \mathbb{R}(\sup(U) < r < \sup(V))$. Since $U < r$, $\neg P(f^{-1}(r))$. Since $r < V$, $P(f^{-1}(r))$. *Contradiction.* Similarly for ii).

iii) If $\sup(U) = \inf(V) = r$, then $P(r)$ and $\neg P(r)$ 5). *Contradiction.*

Argument or *proof*?

P$_1$: $P(f^{-1}(s))$
P$_2$: $\forall\mathcal{X}\subseteq\mathbb{R}(\mathrm{BA}(\mathcal{X})\wedge\forall r(r\in\mathcal{X}\to\varphi(f^{-1}(r)))\to\varphi(f^{-1}(\sup(\mathcal{X}))))$
P$_3$: $\forall\mathcal{X}\subseteq\mathbb{R}(\mathrm{BB}(\mathcal{X})\wedge\forall r(r\in\mathcal{X}\to\varphi(f^{-1}(r)))\to\varphi(f^{-1}(\inf(\mathcal{X}))))$
C: $P(f^{-1}(t))$

How do we get from $\mathrm{P}_1, \mathrm{P}_2, \mathrm{P}_3$ to C? (Reconstructing Weber 2021 based on Chase)

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   - i-ii) $\exists r\in\mathbb{R}(\sup(U)<r<\sup(V))$. Since $U<r$, $\neg P(f^{-1}(r))$. Since $r<V$, $P(f^{-1}(r))$. *Contradiction.* Similarly for ii).
   - iii) If $\sup(U)=\inf(V)=r$, then $P(r)$ and $\neg P(r)$ 5). *Contradiction.*

Reverse philosophical analysis (part 1)

P_0: "Set up" axioms Σ.

P_1: $P(f^{-1}(s))$

P_2: $\forall \mathcal{X} \subseteq \mathbb{R}(\mathrm{BA}(\mathcal{X}) \wedge \forall r(r \in \mathcal{X} \to \varphi(f^{-1}(r))) \to \varphi(f^{-1}(\sup(\mathcal{X}))))$

P_3: $\forall \mathcal{X} \subseteq \mathbb{R}(\mathrm{BB}(\mathcal{X}) \wedge \forall r(r \in \mathcal{X} \to \varphi(f^{-1}(r))) \to \varphi(f^{-1}(\inf(\mathcal{X}))))$

P_4: Extension of $\mathrm{Comprehension}$ to predicates containing P, C, f.

P_5: $\mathrm{Supremum\ Principle}$: $\forall \mathcal{X} \subseteq \mathbb{R}(\mathrm{BA}(\mathcal{X}) \to \sup(\mathcal{X}) \text{ exists})$
$\mathrm{Infimum\ Principle}$: $\forall \mathcal{X} \subseteq \mathbb{R}(\mathrm{BB}(\mathcal{X}) \to \inf(\mathcal{X}) \text{ exists})$

C: $P(f^{-1}(t))$

Complication: $\mathrm{P}_2, \mathrm{P}_3, \mathrm{P}_5$ are *3rd-order* and can't be expressed in $\mathcal{L}_2$.

Work around: Rather than considering $\mathcal{X} \subseteq \mathbb{R}$, we consider doubly indexed sequences $\langle q_{m,n} : m, n \in \mathbb{N}\rangle$ where $r_m = \langle q_{m,n} : n \in \mathbb{N}\rangle$ is s.t. $\forall k \forall i(|q_k - q_{k+i}| \leq 2^{-k})$.

- We consider the "sequential reformulations" of $\mathrm{P}_2, \mathrm{P}_3, \mathrm{P}_5$ in $\mathcal{L}_2$.
  - Call these $\mathrm{LCC}_{\sup}$, $\mathrm{LCC}_{\inf}$, Sup, and Inf.
  - Claim: This does not change the rhetorical setting.

Reverse philosophical analysis (part 2)

- Enthymemic premise of the continuous sorites:
  1) Representation Theorem for $\exists f : C \to [s,t]$ provable in T.
  2) The extension of Comprehension to P, C, f over T.
  3) The Sup and Inf Principles.
- A classical result of Reverse Mathematics:

$$\mathsf{RCA}_0 \vdash \mathrm{Sup} \leftrightarrow \mathrm{Inf} \leftrightarrow \mathrm{ACA}$$

(ACA is Arithmetical Comprehension formulated as a single $\mathcal{L}_2$-sentence.)

- ACA is (in some sense) *powerful* – e.g.
  - $\mathrm{ACA}_0 = \mathrm{RCA}_0 + \mathrm{ACA} \vdash$ The Halting Problem (K) exists.
  - For all $n \in \mathbb{N}$, $\mathrm{ACA}_0 \vdash K^{(n)}$ exists.
  - ACA is ***denied*** by constructivists and some predicativists.
- Broaching the ***second level of mathematical involvement***:
  - The reasoning of the continuous sorites requires *both mathematical representation* using $\mathbb{R}$ *and* the assumption that $\mathbb{R}$ satisfies properties ***beyond its definition***.

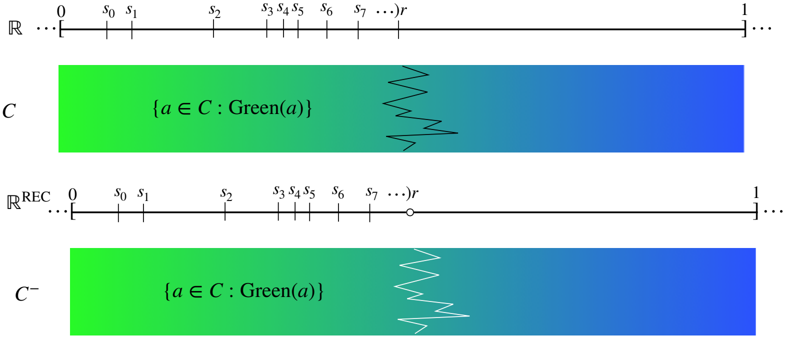
Specker sequences and the continuous sorites (part 1)

- The Carrot: Novel rhetorical options
  1) *Reject* Representation Theorem or (better?) T.
  2) *Reject* extension of Comprehension to empirical vocabulary.
  3) *Reject* ACA.
- The Stick: In order to monger the paradox, either you or your customers have to *accept* ACA.
  - Or: It's hard to do "fancy analytic philosophy" and be a nominalist/fictionalist/constructivist *at the same time*.
- What is at issue can be illustrated by a ***Specker sequence***.
  - I.e. $S = \langle s_k \in \mathbb{Q} : k \in \mathbb{N} \rangle$ with the following properties:
    i) computable: there is an algorithm $\alpha(i) = s_i$
    ii) monotonic: $i < j \rightarrow s_i < s_j$
    iii) bounded: in fact $\forall i(s_i < 1)$
    iv) non-computable least upper bound: $r = \sup(\{s_k : k \in \mathbb{N}\})$ is not given by a computable Cauchy sequence (i.e. no computable modulus of convergence)

Specker sequences and the continuous sorites (part 2)

- Definition of $S = \langle s_k \in \mathbb{Q} : k \in \mathbb{N}\rangle$:
  - Let $\alpha(x)$ be a computable injective enumeration of K – i.e. $K = \{\alpha(0), \alpha(1), \alpha(2), \ldots\}$ without repetitions.
  - Let
  $$s_k = \Sigma_{i=0}^{k} 2^{-\alpha(i)-1} < 1$$
  - If we let $r = \sup(\langle s_k \rangle)$ and assume that $r = \langle q_k \rangle$ with $\forall k \forall i (|q_k - q_{k+i}|) \leq 2^{-k}$ (e.g.) then we could decide K.
- Measurement-theoretic operationalization:
  - Think of $\alpha(x)$ as outputting digits in the expansion of $r = \sup\{g(a) : a \in C \wedge \text{Green}(a)\}$ and $g(x)$ scales C into $[0, 1]$.
    - If there is i s.t. $\alpha(i) = m$, then the mth digit of r is 1.
    - And thus $\text{Green}(a)$ for all $a \in C$ s.t. $g(a) \leq 2^{m-1}$.
  - Must we also have $\text{Green}(g^{-1}(r))$?
    - Weber & Colyvan: Yes (stipulatively) if we presume that $\text{Green}(x)$ is within the scope of LCC.
    - Empirically: No (presumably).
- Q: Are the boundaries of vague predicates *regular*/*predictable* or *chaotic*/*unknowable*?

The recursive reals and the missing shade of blue (cf. Hume 1748)



Refined rhetorical options:

- Realism about $\mathbb{R}$ / Continuism about C: Reject the LCC.
- Constructivism about $\mathbb{R}$ / Discontinuism about C: I.e. "gaps" in both; so reject both mathematical and empirical sups/infs.
- Realism about $\mathbb{R}$ / Discontinuism about C: $f(r) \notin C$ (totality fails); accept mathematical sups/infs; reject empirical sups/infs.
- Constructivism about $\mathbb{R}$ / Continuism about C: $f^{-1}(b) \notin \mathbb{R}^{\mathrm{REC}}$: extension of schema to C, f, P "smuggles in" Σ^0_1-comprehension.

A two way street?

- What I'm ***not*** claiming:
  - The continuous sorites poses a natural or important problem.
  - Any of the above is the *best* response.
  - (If you don't like this example of "reverse philosophy", we've got many others.)

$\Leftarrow$ What I ***am*** claiming:
  - The continuous sorites is a good example of how contemporary analytic philosophy is often mathematically involved.
  - Reverse mathematics helps us characterize the involvement and makes available novel rhetorical options.

$\Rightarrow$ Novel (?) mathematical questions:

1) Does Hölder's Theorem$^{-,\pm}$ imply Σ^0_1-induction over RCA$_0^*$?
   - Relationship to Simpson & Yokoyama 2012.
2) What is the best way to formalize Hölder's Theorem in $\mathcal{L}_2$?
   - Status of 'empirical comprehension'? ACA_0 or Z_2 over RCA$_0$?
3) Strength of other rep. thms – e.g. Debreu (1954/59).
4) Reverse mathematics of connected spaces.
   - Relationship to Mummert 2005, Walker 2008, Dhzafarov 2019.

Q: Who *owns* the sorites?

Philosophical Logic (e.g. non-classical, …)	Philosophy of language & mind, …	Linguistics (semantics)	Empirical Psychology (psychophysics, …)	Mathematical Psychology (measurement, …)	Economics (decision theory, …)	Mathematics (logic, topology, …)
Weber 2021 Magidor 2012 Corbreros et al. 2012 Weber & Colyvan 2010 Field 2003/8 Boolos 1991 Parikh 1983 van Frassen 1966 Zadeh 1965 Körner 1955 Halldén 1949	Pagin 2011 Hyde 2008 Shapiro 2006 Graff 2001 Keefe 2000 Smith & Keefe 2000 Williamson 1994 Sorenson 1988 Fine 1975 Wright 1975 Dummett 1972/5 Waismann 1945 Black 1937 Russell 1923 Diogenes Laertius c230 CE Eubulides c350 BCE	Itzakhi 2021 Burnett 2017 Lassiter 2017 van Rooij 2011 Sassoon 2010 Kennedy 2007 Barker 2002 Kamp1975	Cervantes & E. Dzhafarov, 2019. Gescheider 1997 Krantz 1964 Stevens 1946 Wright & Pitt 1934 Weber 1830	E. Dzhafarov, & Colonius 2022 Batchelder et. al 2016/18 Dzhafarov & Dzhafarov 2010a/b Fishburn 1973 Krantz et. 1971… Tversky 1967 Luce et al. 1963 von Helmholtz 18996	Anand 1993 Quinn 1990 Quinn 1987 Kahneman & Tversky 1979 Debreu 1959 Scott & Suppes 1958 Luce 1958 Armstrong 1948 Borel 1907	Dzhafarov 2019 Dean 2018 Hájek 2013 Sazovon 1995 Nelson, Buss 1986 Vopenka 1979 Cook 1975 Parikh 1971 Yessenin-Volpin 1961/72 Wang 1958 Borel 1952 Bernays 1935 Fréchet 1913 Frege 1879

A: Mathematics.