Plato, Brouwer, and classification

Sam Sanders (jww Dag Normann)

Dept. of Mathematics, TU Darmstadt, Germany

Logic Colloquium, UConn, Oct. 9, 2020

In a nutshell

During this talk, I will (not necessarily in that order)...

In a nutshell

During this talk, I will (not necessarily in that order)...

- gradually introduce the program Reverse Mathematics (RM) with an eye on philosophical/foundational questions,

In a nutshell

During this talk, I will (not necessarily in that order)...

- gradually introduce the program Reverse Mathematics (RM) with an eye on philosophical/foundational questions,
- present some recent RM results that are jww Dag Normann,

In a nutshell

During this talk, I will (not necessarily in that order)...

- gradually introduce the program Reverse Mathematics (RM) with an eye on philosophical/foundational questions,
- present some recent RM results that are jww Dag Normann,
- and discuss the relevance to philosophy and foundations of mathematics.

In a nutshell

During this talk, I will (not necessarily in that order)...

- gradually introduce the program Reverse Mathematics (RM) with an eye on philosophical/foundational questions,
- present some recent RM results that are jww Dag Normann,
- and discuss the relevance to philosophy and foundations of mathematics.

My collaborators are not guilty of my opinions.

The aim of RM is classification

To find the minimal axioms necessary for proving a theorem of ordinary mathematics.

The aim of RM is classification

To find the minimal axioms necessary for proving a theorem of ordinary mathematics.

(Q1) What does ordinary mathematics mean?

The aim of RM is classification

To find the minimal axioms necessary for proving a theorem of ordinary mathematics.

(Q1) What does ordinary mathematics mean?

(Q2) What scale does ‘minimal’ refer to and why choose that one?

The aim of RM is classification

To find the minimal axioms necessary for proving a theorem of ordinary mathematics.

(Q1) What does ordinary mathematics mean?

(Q2) What scale does ‘minimal’ refer to and why choose that one?

(Q3) Are ‘the’ minimal axioms always unique and unambiguous?

The aim of RM is classification

To find the minimal axioms necessary for proving a theorem of ordinary mathematics.

(Q1) What does ordinary mathematics mean?

(Q2) What scale does 'minimal' refer to and why choose that one?

(Q3) Are 'the' minimal axioms always unique and unambiguous?

For Part I, it suffices to know that there are three major classes, weak, medium, and strong, of logical strength (Gödel hierarchy).

The aim of RM is classification

To find the minimal axioms necessary for proving a theorem of ordinary mathematics.

(Q1) What does ordinary mathematics mean?

(Q2) What scale does 'minimal' refer to and why choose that one?

(Q3) Are 'the' minimal axioms always unique and unambiguous?

For Part I, it suffices to know that there are three major classes, weak, medium, and strong, of logical strength (Gödel hierarchy).

The investigation of RM generally takes place in the weak part and the 'lower end' of the medium part, using the language L_2 of second-order arithmetic Z_2.

The aim of RM is classification

To find the minimal axioms necessary for proving a theorem of ordinary mathematics.

(Q1) What does ordinary mathematics mean?

(Q2) What scale does ‘minimal’ refer to and why choose that one?

(Q3) Are ‘the’ minimal axioms always unique and unambiguous?

For Part I, it suffices to know that there are three major classes, weak, medium, and strong, of logical strength (Gödel hierarchy).

The investigation of RM generally takes place in the weak part and the ‘lower end’ of the medium part, using the language L_2 of second-order arithmetic Z_2.

The language L_2 only includes first and second-order variables ‘$n \in \mathbb{N}$’ and ‘$X \subseteq \mathbb{N}$’. Higher-order objects are represented/coded via the latter.

The aim of RM is classification

To find the minimal axioms necessary for proving a theorem of ordinary mathematics.

(Q1) What does ordinary mathematics mean?

(Q2) What scale does 'minimal' refer to and why choose that one?

(Q3) Are 'the' minimal axioms always unique and unambiguous?

For Part I, it suffices to know that there are three major classes, weak, medium, and strong, of logical strength (Gödel hierarchy).

The investigation of RM generally takes place in the weak part and the 'lower end' of the medium part, using the language L_2 of second-order arithmetic Z_2.

The language L_2 only includes first and second-order variables '$n \in \mathbb{N}$' and '$X \subseteq \mathbb{N}$'. Higher-order objects are represented/coded via the latter. Any formalisation involves representations/codes.

Q1: What is ordinary mathematics?

In SOSOA, the Bible of RM, Simpson describes ordinary mathematics as:

Q1: What is ordinary mathematics?

In SOSOA, the Bible of RM, Simpson describes ordinary mathematics as:

> *that body of mathematics that is prior to or independent of the introduction of abstract set theoretic concepts*.

Q1: What is ordinary mathematics?

In SOSOA, the Bible of RM, Simpson describes ordinary mathematics as:

> *that body of mathematics that is prior to or independent of the introduction of abstract set theoretic concepts.*

with the caveat that theorems should not be modified:

Q1: What is ordinary mathematics?

In SOSOA, the Bible of RM, Simpson describes ordinary mathematics as:

> *that body of mathematics that is prior to or independent of the introduction of abstract set theoretic concepts.*

with the caveat that theorems should not be modified:

> *The typical constructivist response to a nonconstructive mathematical theorem is to modify the theorem by adding hypotheses or "extra data". In contrast, our approach in this book is to analyze the provability of mathematical theorems as they stand, passing to stronger subsystems of Z_2 if necessary.* (SOSOA, p. 32)

Q1: What is ordinary mathematics?

In SOSOA, the Bible of RM, Simpson describes ordinary mathematics as:

> *that body of mathematics that is prior to or independent of the introduction of abstract set theoretic concepts.*

with the caveat that theorems should not be modified:

> *The typical constructivist response to a nonconstructive mathematical theorem is to modify the theorem by adding hypotheses or "extra data". In contrast, our approach in this book is to analyze the provability of mathematical theorems as they stand, passing to stronger subsystems of Z_2 if necessary.* (SOSOA, p. 32)

The final sentence is somewhat paradoxical as follows.

Coding ordinary mathematics

All here shall known ε-δ-continuity for $f : [0,1] \to \mathbb{R}$ as follows:

$$(\forall \varepsilon > 0, x \in [0,1])(\exists \delta > 0)(\forall y \in [0,1])(|x-y| < \delta \to |f(x)-f(y)| < \varepsilon).$$

Coding ordinary mathematics

All here shall known ε-δ-continuity for $f : [0,1] \to \mathbb{R}$ as follows:

$$(\forall \varepsilon > 0, x \in [0,1])(\exists \delta > 0)(\forall y \in [0,1])(|x-y| < \delta \to |f(x)-f(y)| < \varepsilon).$$

Now compare this to 'continuity-via-codes' in L_2 from SOSOA:

II.6. CONTINUOUS FUNCTIONS 85

DEFINITION II.6.1 (continuous functions). Within RCA_0, let $\widehat{A}$ and $\widehat{B}$ be complete separable metric spaces. A (code for a) *continuous partial function* ϕ from $\widehat{A}$ to $\widehat{B}$ is a set of quintuples $\Phi \subseteq \mathbb{N} \times A \times \mathbb{Q}^+ \times B \times \mathbb{Q}^+$ which is required to have certain properties. We write $(a,r)\Phi(b,s)$ as an abbreviation for $\exists n\,((n,a,r,b,s) \in \Phi)$. The properties which we require are:

1. if $(a,r)\Phi(b,s)$ and $(a,r)\Phi(b',s')$, then $d(b,b') \le s + s'$;
2. if $(a,r)\Phi(b,s)$ and $(a',r') < (a,r)$, then $(a',r')\Phi(b,s)$;
3. if $(a,r)\Phi(b,s)$ and $(b,s) < (b',s')$, then $(a,r)\Phi(b',s')$;

where the notation $(a',r') < (a,r)$ means that $d(a,a') + r' < r$.

Coding ordinary mathematics

All here shall known ε-δ-continuity for $f : [0,1] \to \mathbb{R}$ as follows:

$$(\forall \varepsilon > 0, x \in [0,1])(\exists \delta > 0)(\forall y \in [0,1])(|x-y| < \delta \to |f(x)-f(y)| < \varepsilon).$$

Now compare this to 'continuity-via-codes' in L_2 from SOSOA:

II.6. CONTINUOUS FUNCTIONS 85

DEFINITION II.6.1 (continuous functions). Within RCA_0, let $\widehat{A}$ and $\widehat{B}$ be complete separable metric spaces. A (code for a) *continuous partial function* ϕ from $\widehat{A}$ to $\widehat{B}$ is a set of quintuples $\Phi \subseteq \mathbb{N} \times A \times \mathbb{Q}^+ \times B \times \mathbb{Q}^+$ which is required to have certain properties. We write $(a,r)\Phi(b,s)$ as an abbreviation for $\exists n\,((n,a,r,b,s) \in \Phi)$. The properties which we require are:

1. if $(a,r)\Phi(b,s)$ and $(a,r)\Phi(b',s')$, then $d(b,b') \le s+s'$;
2. if $(a,r)\Phi(b,s)$ and $(a',r') < (a,r)$, then $(a',r')\Phi(b,s)$;
3. if $(a,r)\Phi(b,s)$ and $(b,s) < (b',s')$, then $(a,r)\Phi(b',s')$;

where the notation $(a',r') < (a,r)$ means that $d(a,a') + r' < r$.

Problems:

1. NO mainstream math textbook uses Def. II.6.1 above.

Coding ordinary mathematics

All here shall known ε-δ-continuity for $f : [0,1] \to \mathbb{R}$ as follows:

$$(\forall \varepsilon > 0, x \in [0,1])(\exists \delta > 0)(\forall y \in [0,1])(|x-y| < \delta \to |f(x)-f(y)| < \varepsilon).$$

Now compare this to 'continuity-via-codes' in L_2 from SOSOA:

II.6. CONTINUOUS FUNCTIONS 85

DEFINITION II.6.1 (continuous functions). Within RCA_0, let $\widehat{A}$ and $\widehat{B}$ be complete separable metric spaces. A (code for a) *continuous partial function* ϕ from $\widehat{A}$ to $\widehat{B}$ is a set of quintuples $\Phi \subseteq \mathbb{N} \times A \times \mathbb{Q}^+ \times B \times \mathbb{Q}^+$ which is required to have certain properties. We write $(a,r)\Phi(b,s)$ as an abbreviation for $\exists n\,((n,a,r,b,s) \in \Phi)$. The properties which we require are:

1. if $(a,r)\Phi(b,s)$ and $(a,r)\Phi(b',s')$, then $d(b,b') \leq s+s'$;
2. if $(a,r)\Phi(b,s)$ and $(a',r') < (a,r)$, then $(a',r')\Phi(b,s)$;
3. if $(a,r)\Phi(b,s)$ and $(b,s) < (b',s')$, then $(a,r)\Phi(b',s')$;

where the notation $(a',r') < (a,r)$ means that $d(a,a') + r' < r$.

Problems:

1. NO mainstream math textbook uses Def. II.6.1 above.
2. Using Def. II.6.1 introduces a modulus of continuity, typical "extra data" from constructive math (Kohlenbach).

Coding ordinary mathematics

All here shall known ε-δ-continuity for $f : [0,1] \to \mathbb{R}$ as follows:

$$(\forall \varepsilon > 0, x \in [0,1])(\exists \delta > 0)(\forall y \in [0,1])(|x-y| < \delta \to |f(x)-f(y)| < \varepsilon).$$

Now compare this to 'continuity-via-codes' in L_2 from SOSOA:

II.6. CONTINUOUS FUNCTIONS 85

DEFINITION II.6.1 (continuous functions). Within RCA_0, let $\widehat{A}$ and $\widehat{B}$ be complete separable metric spaces. A (code for a) *continuous partial function* ϕ from $\widehat{A}$ to $\widehat{B}$ is a set of quintuples $\Phi \subseteq \mathbb{N} \times A \times \mathbb{Q}^+ \times B \times \mathbb{Q}^+$ which is required to have certain properties. We write $(a,r)\Phi(b,s)$ as an abbreviation for $\exists n\,((n,a,r,b,s) \in \Phi)$. The properties which we require are:

1. if $(a,r)\Phi(b,s)$ and $(a,r)\Phi(b',s')$, then $d(b,b') \leq s + s'$;
2. if $(a,r)\Phi(b,s)$ and $(a',r') < (a,r)$, then $(a',r')\Phi(b,s)$;
3. if $(a,r)\Phi(b,s)$ and $(b,s) < (b',s')$, then $(a,r)\Phi(b',s')$;

where the notation $(a',r') < (a,r)$ means that $d(a,a') + r' < r$.

Problems:

1. NO mainstream math textbook uses Def. II.6.1 above.
2. Using Def. II.6.1 introduces a modulus of continuity, typical "extra data" from constructive math (Kohlenbach).

Question: why does Def. II.6.1 still count as ordinary math?

Coding ordinary mathematics

Question: why does Def. II.6.1 still count as ordinary math?

Coding ordinary mathematics

Question: why does Def. II.6.1 still count as ordinary math?

Based on a construction by D. Normann, U. Kohlenbach shows that these two definitions are equivalent in a weak higher-order system based on the well-known *weak König's lemma* (WKL).

$$(\forall \varepsilon > 0, x \in [0,1])(\exists \delta > 0)(\forall y \in [0,1])(|x-y| < \delta \to |f(x)-f(y)| < \varepsilon).$$

II.6. CONTINUOUS FUNCTIONS 85

DEFINITION II.6.1 (continuous functions). Within RCA_0, let $\widehat{A}$ and $\widehat{B}$ be complete separable metric spaces. A (code for a) *continuous partial function* ϕ from $\widehat{A}$ to $\widehat{B}$ is a set of quintuples $\Phi \subseteq \mathbb{N} \times A \times \mathbb{Q}^+ \times B \times \mathbb{Q}^+$ which is required to have certain properties. We write $(a,r)\Phi(b,s)$ as an abbreviation for $\exists n\,((n,a,r,b,s) \in \Phi)$. The properties which we require are:

1. if $(a,r)\Phi(b,s)$ and $(a,r)\Phi(b',s')$, then $d(b,b') \leq s+s'$;
2. if $(a,r)\Phi(b,s)$ and $(a',r') < (a,r)$, then $(a',r')\Phi(b,s)$;
3. if $(a,r)\Phi(b,s)$ and $(b,s) < (b',s')$, then $(a,r)\Phi(b',s')$;

where the notation $(a',r') < (a,r)$ means that $d(a,a') + r' < r$.

Coding ordinary mathematics

Question: why does Def. II.6.1 still count as ordinary math?

Based on a construction by D. Normann, U. Kohlenbach shows that these two definitions are equivalent in a weak higher-order system based on the well-known *weak König's lemma* (WKL).

$$(\forall \varepsilon > 0, x \in [0,1])(\exists \delta > 0)(\forall y \in [0,1])(|x-y| < \delta \rightarrow |f(x)-f(y)| < \varepsilon).$$

II.6. CONTINUOUS FUNCTIONS 85

DEFINITION II.6.1 (continuous functions). Within RCA_0, let $\widehat{A}$ and $\widehat{B}$ be complete separable metric spaces. A (code for a) *continuous partial function* ϕ from $\widehat{A}$ to $\widehat{B}$ is a set of quintuples $\Phi \subseteq \mathbb{N} \times A \times \mathbb{Q}^+ \times B \times \mathbb{Q}^+$ which is required to have certain properties. We write $(a, r)\Phi(b, s)$ as an abbreviation for $\exists n\,((n, a, r, b, s) \in \Phi)$. The properties which we require are:

1. if $(a, r)\Phi(b, s)$ and $(a, r)\Phi(b', s')$, then $d(b, b') \leq s + s'$;
2. if $(a, r)\Phi(b, s)$ and $(a', r') < (a, r)$, then $(a', r')\Phi(b, s)$;
3. if $(a, r)\Phi(b, s)$ and $(b, s) < (b', s')$, then $(a, r)\Phi(b', s')$;

where the notation $(a', r') < (a, r)$ means that $d(a, a') + r' < r$.

Problem solved: using codes as in Def. II.6.1 or plain ε-δ-continuity yields the 'same theorems', assuming WKL.

Coding ordinary mathematics

Around 1850, Riemann's *Habilschrift* introduces his integral and forces discontinuous functions into mainstream math.

Coding ordinary mathematics

Around 1850, Riemann's *Habilschrift* introduces his integral and forces discontinuous functions into mainstream math.

Theorem (Arzela, 1885)

Let $f_n : ([0,1] \times \mathbb{N}) \to \mathbb{R}$ *be a sequence such that*

1. *Each* f_n *is Riemann integrable on* $[0,1]$.
2. *There is* $M > 0$ *such that* $(\forall n \in \mathbb{N}, x \in [0,1])(|f_n(x)| \leq M)$.
3. $\lim_{n\to\infty} f_n = f$ *exists and is Riemann integrable.*

Then $\lim_{n\to\infty} \int_0^1 f_n(x)dx = \int_0^1 f(x)dx$.

Coding ordinary mathematics

Around 1850, Riemann's *Habilschrift* introduces his integral and forces discontinuous functions into mainstream math.

Theorem (Arzela, 1885)

Let $f_n : ([0,1] \times \mathbb{N}) \to \mathbb{R}$ *be a sequence such that*

1. *Each* f_n *is Riemann integrable on* $[0,1]$.
2. *There is* $M > 0$ *such that* $(\forall n \in \mathbb{N}, x \in [0,1])(|f_n(x)| \leq M)$.
3. $\lim_{n\to\infty} f_n = f$ *exists and is Riemann integrable.*

Then $\lim_{n\to\infty} \int_0^1 f_n(x)dx = \int_0^1 f(x)dx$.

Formulated with codes in L_2, this theorem falls in the 'weak' range.

Coding ordinary mathematics

Around 1850, Riemann's *Habilschrift* introduces his integral and forces discontinuous functions into mainstream math.

Theorem (Arzela, 1885)

Let $f_n : ([0,1] \times \mathbb{N}) \to \mathbb{R}$ *be a sequence such that*

1. *Each* f_n *is Riemann integrable on* $[0,1]$.
2. *There is* $M > 0$ *such that* $(\forall n \in \mathbb{N}, x \in [0,1])(|f_n(x)| \leq M)$.
3. $\lim_{n\to\infty} f_n = f$ *exists and is Riemann integrable.*

Then $\lim_{n\to\infty} \int_0^1 f_n(x)dx = \int_0^1 f(x)dx$.

Formulated with codes in L_2, this theorem falls in the 'weak' range.

Formulated without codes, this theorem is at the very top of the 'medium' range (near Z_2), far beyond the usual range of RM.

Coding ordinary mathematics

Around 1850, Riemann's *Habilschrift* introduces his integral and forces discontinuous functions into mainstream math.

Theorem (Arzela, 1885)

Let $f_n : ([0,1] \times \mathbb{N}) \to \mathbb{R}$ *be a sequence such that*

1. *Each* f_n *is Riemann integrable on* $[0,1]$.
2. *There is* $M > 0$ *such that* $(\forall n \in \mathbb{N}, x \in [0,1])(|f_n(x)| \leq M)$.
3. $\lim_{n\to\infty} f_n = f$ *exists and is Riemann integrable.*

Then $\lim_{n\to\infty} \int_0^1 f_n(x)dx = \int_0^1 f(x)dx$.

Formulated with codes in L_2, this theorem falls in the 'weak' range.

Formulated without codes, this theorem is at the very top of the 'medium' range (near Z_2), far beyond the usual range of RM.

See arxiv: Normann-Sanders, On the uncountability of $\mathbb{R}$.

Intermediate conclusion

Intermediate conclusion

If one wishes to study mathematical theorems as they stand, coding in L_2 plays the following role:

Intermediate conclusion

If one wishes to study mathematical theorems as they stand, coding in L_2 plays the following role:

Coding continuous functions in L_2 is OK, following the work of Normann and Kohlenbach.

Intermediate conclusion

If one wishes to study mathematical theorems as they stand, coding in L_2 plays the following role:

Coding continuous functions in L_2 is OK, following the work of Normann and Kohlenbach.

Coding Riemann integrable functions (=continuous AE and bounded) in L_2 is not OK, following the work of Normann-Sanders.

Intermediate conclusion

If one wishes to study mathematical theorems as they stand, coding in L_2 plays the following role:

Coding continuous functions in L_2 is OK, following the work of Normann and Kohlenbach.

Coding Riemann integrable functions (=continuous AE and bounded) in L_2 is not OK, following the work of Normann-Sanders.

The difference between ‘codes’ or ‘no codes’ for Riemann integrable functions can be huge, as shown by Arzela’s convergence theorem.

Intermediate conclusion

If one wishes to study mathematical theorems as they stand, coding in L_2 plays the following role:

Coding continuous functions in L_2 is OK, following the work of Normann and Kohlenbach.

Coding Riemann integrable functions (=continuous AE and bounded) in L_2 is not OK, following the work of Normann-Sanders.

The difference between 'codes' or 'no codes' for Riemann integrable functions can be huge, as shown by Arzela's convergence theorem.

To properly study discontinuous functions, Kohlenbach has proposed higher-order RM involving all finite types.

Intermediate conclusion

If one wishes to study mathematical theorems as they stand, coding in L_2 plays the following role:

Coding continuous functions in L_2 is OK, following the work of Normann and Kohlenbach.

Coding Riemann integrable functions (=continuous AE and bounded) in L_2 is not OK, following the work of Normann-Sanders.

The difference between 'codes' or 'no codes' for Riemann integrable functions can be huge, as shown by Arzela's convergence theorem.

To properly study discontinuous functions, Kohlenbach has proposed higher-order RM involving all finite types. The language L_ω has variables for

$$n \in \mathbb{N}, f : \mathbb{N} \to \mathbb{N}, Y : \mathbb{N}^{\mathbb{N}} \to \mathbb{N}, F : \mathbb{R} \to \mathbb{R}, G : (\mathbb{R} \to \mathbb{R}) \to \mathbb{R}, \dots$$

Intermediate conclusion

If one wishes to study mathematical theorems as they stand, coding in L_2 plays the following role:

Coding continuous functions in L_2 is OK, following the work of Normann and Kohlenbach.

Coding Riemann integrable functions (=continuous AE and bounded) in L_2 is not OK, following the work of Normann-Sanders.

The difference between 'codes' or 'no codes' for Riemann integrable functions can be huge, as shown by Arzela's convergence theorem.

To properly study discontinuous functions, Kohlenbach has proposed higher-order RM involving all finite types. The language L_ω has variables for

$$n \in \mathbb{N}, f : \mathbb{N} \to \mathbb{N}, Y : \mathbb{N}^{\mathbb{N}} \to \mathbb{N}, F : \mathbb{R} \to \mathbb{R}, G : (\mathbb{R} \to \mathbb{R}) \to \mathbb{R}, \dots$$

Higher-order RM is not the full answer, as our answer to Q3 shows.

Q3: are ‘the’ minimal axioms always unique?

Q3: are ‘the’ minimal axioms always unique?

Tao and others have stressed the intimate connection between ‘compactness’ and ‘local-global principles’.

Q3: are 'the' minimal axioms always unique?

Tao and others have stressed the intimate connection between 'compactness' and 'local-global principles'.

PIT_o is one of the first 'local-global principles'.

Theorem (PIT_o, Pincherle, 1885)

A locally bounded function on $2^{\mathbb{N}}$ is bounded.

Q3: are 'the' minimal axioms always unique?

Tao and others have stressed the intimate connection between 'compactness' and 'local-global principles'.

PIT_o is one of the first 'local-global principles'.

Theorem (PIT_o, Pincherle, 1885)

A locally bounded function on $2^{\mathbb{N}}$ is bounded.

Pincherle stresses that his theorem applies to any function.

Q3: are 'the' minimal axioms always unique?

Tao and others have stressed the intimate connection between 'compactness' and 'local-global principles'.

PIT_o is one of the first 'local-global principles'.

Theorem (PIT_o, Pincherle, 1885)

A locally bounded function on $2^{\mathbb{N}}$ is bounded.

Pincherle stresses that his theorem applies to any function.

1. Assuming a fragment of countable choice, we have $\mathsf{WKL} \leftrightarrow \mathsf{PIT}_o$, i.e. PIT_o is in the weak range.

Q3: are 'the' minimal axioms always unique?

Tao and others have stressed the intimate connection between 'compactness' and 'local-global principles'.

PIT_o is one of the first 'local-global principles'.

Theorem (PIT_o, Pincherle, 1885)

A locally bounded function on $2^{\mathbb{N}}$ is bounded.

Pincherle stresses that his theorem applies to any function.

1. Assuming a fragment of countable choice, we have $\text{WKL} \leftrightarrow \text{PIT}_o$, i.e. PIT_o is in the weak range.
2. Without countable choice, PIT_o cannot be proved in the medium range (but provable without AC).

Q3: are ‘the’ minimal axioms always unique?

Tao and others have stressed the intimate connection between ‘compactness’ and ‘local-global principles’.

PIT_o is one of the first ‘local-global principles’.

Theorem (PIT_o, Pincherle, 1885)

A locally bounded function on $2^{\mathbb{N}}$ is bounded.

Pincherle stresses that his theorem applies to any function.

1. Assuming a fragment of countable choice, we have $\mathsf{WKL} \leftrightarrow \mathsf{PIT}_o$, i.e. PIT_o is in the weak range.
2. Without countable choice, PIT_o cannot be proved in the medium range (but provable without AC).

No unique/unambiguous minimal collection of axioms!

Q3: are ‘the’ minimal axioms always unique?

Q3: are 'the' minimal axioms always unique?

Pincherle's theorem PIT_o is just one example. Open sets give rise to many examples.

Q3: are 'the' minimal axioms always unique?

Pincherle's theorem PIT_o is just one example. Open sets give rise to many examples.

In RM, an open set is given by a union of basic open balls $\cup_{n\in\mathbb{N}}(a_n, b_n)$.

Q3: are 'the' minimal axioms always unique?

Pincherle's theorem PIT_o is just one example. Open sets give rise to many examples.

In RM, an open set is given by a union of basic open balls $\cup_{n\in\mathbb{N}}(a_n, b_n)$.

Following Kreuzer and others, we have studied open sets in $\mathbb{R}$ via (third-order) characteristic functions.

Q3: are 'the' minimal axioms always unique?

Pincherle's theorem PIT_o is just one example. Open sets give rise to many examples.

In RM, an open set is given by a union of basic open balls $\cup_{n\in\mathbb{N}}(a_n, b_n)$.

Following Kreuzer and others, we have studied open sets in $\mathbb{R}$ via (third-order) characteristic functions. The following thms then behave in the same way as PIT_o:

1. Urysohn lemma
2. Tietze extension theorem
3. Cantor-Bendixson theorem
4. Baire-Category theorem
5. ...

Intermediate conclusion II

Our answers to Q1 and Q3 have yielded the following:

Intermediate conclusion II

Our answers to Q1 and Q3 have yielded the following:

Intermediate conclusion II

Our answers to Q1 and Q3 have yielded the following:

Coding in L_2 can change the logical strength of thms involving Riemann integrable functions, unacceptable from the pov of RM.

Intermediate conclusion II

Our answers to Q1 and Q3 have yielded the following:

Coding in L_2 can change the logical strength of thms involving Riemann integrable functions, unacceptable from the pov of RM.

Switching to L_ω and Kohlenbach's higher-order RM seems to create other problems involving minimal axioms and countable choice.

Intermediate conclusion II

Our answers to Q1 and Q3 have yielded the following:

Coding in L_2 can change the logical strength of thms involving Riemann integrable functions, unacceptable from the pov of RM.

Switching to L_ω and Kohlenbach's higher-order RM seems to create other problems involving minimal axioms and countable choice.

Our hubris: everything seems wrong about RM.

Intermediate conclusion II

Our answers to Q1 and Q3 have yielded the following:

Coding in L_2 can change the logical strength of thms involving Riemann integrable functions, unacceptable from the pov of RM.

Switching to L_ω and Kohlenbach's higher-order RM seems to create other problems involving minimal axioms and countable choice.

Our hubris: everything seems wrong about RM.

Our catharsis: the answer to Q2 shows that all these problems go away.

Intermediate conclusion II

Our answers to Q1 and Q3 have yielded the following:

Coding in L_2 can change the logical strength of thms involving Riemann integrable functions, unacceptable from the pov of RM.

Switching to L_ω and Kohlenbach's higher-order RM seems to create other problems involving minimal axioms and countable choice.

Our hubris: everything seems wrong about RM.

Our catharsis: the answer to Q2 shows that all these problems go away.

The aim of RM is: to find the minimal axioms necessary for proving a theorem of ordinary mathematics.

(Q2) What scale does 'minimal' refer to and why choose that one?

Gödel hierarchy

strong
- $\vdots$
- large cardinals
- $\vdots$
- ZFC
- ZC (Zermelo set theory)
- simple type theory

medium
- Z_2 (second-order arithmetic)
- $\vdots$
- Π^1_2-CA_0 (comprehension for Π^1_2-formulas)
- Π^1_1-CA_0 (comprehension for Π^1_1-formulas)
- ATR_0 (arithmetical transfinite recursion)
- ACA_0 (arithmetical comprehension)

weak
- WKL_0 (weak König's lemma)
- RCA_0 (recursive comprehension)
- PRA (primitive recursive arithmetic)
- bounded arithmetic

It is striking that a great many foundational theories are linearly ordered by [consistency strength] $<$. Of course it is possible to construct pairs of artificial theories which are incomparable under $<$. However, this is not the case for the "natural" or non-artificial theories which are usually regarded as significant in the foundations of mathematics.

(Simpson, Gödel Centennial Volume; also: Koelner, Burgess, Friedman,. . .)

Gödel hierarchy

= ‘comprehension’ hierarchy

↑

MORE sets exist

. . .

LESS sets exist

↓

strong

- ⋮
- large cardinals
- ⋮
- ZFC
- ZC (Zermelo set theory)
- simple type theory

medium

- Z_2 (second-order arithmetic)
- ⋮
- Π^1_2-CA_0 (comprehension for Π^1_2-formulas)
- Π^1_1-CA_0 (comprehension for Π^1_1-formulas)
- ATR_0 (arithmetical transfinite recursion)
- ACA_0 (arithmetical comprehension)

weak

- WKL_0 (weak König’s lemma)
- RCA_0 (recursive comprehension)
- PRA (primitive recursive arithmetic)
- bounded arithmetic

Gödel hierarchy

strong
- ⋮
- large cardinals
- ⋮
- ZFC
- ZC (Zermelo set theory)
- simple type theory

Zermelo-Fraenkel set theory with choice
aka 'the' foundation of mathematics

medium
- Z_2 (second-order arithmetic)
- ⋮
- Π^1_2-CA_0 (comprehension for Π^1_2-formulas)
- Π^1_1-CA_0 (comprehension for Π^1_1-formulas)
- ATR_0 (arithmetical transfinite recursion)
- ACA_0 (arithmetical comprehension)

weak
- WKL_0 (weak König's lemma)
- RCA_0 (recursive comprehension)
- PRA (primitive recursive arithmetic)
- bounded arithmetic

Gödel hierarchy

Zermelo-Fraenkel set theory with choice
aka ‘the’ foundation of mathematics

Hilbert-Bernays's Grundlagen der Mathematik

strong
- ⋮
- large cardinals
- ⋮
- ZFC
- ZC (Zermelo set theory)
- simple type theory

medium
- Z_2 (second-order arithmetic)
- ⋮
- Π^1_2-CA_0 (comprehension for Π^1_2-formulas)
- Π^1_1-CA_0 (comprehension for Π^1_1-formulas)
- ATR_0 (arithmetical transfinite recursion)
- ACA_0 (arithmetical comprehension)

weak
- WKL_0 (weak König's lemma)
- RCA_0 (recursive comprehension)
- PRA (primitive recursive arithmetic)
- bounded arithmetic

Gödel hierarchy

	strong	⋮ large cardinals ⋮ ZFC ZC (Zermelo set theory) simple type theory
Zermelo-Fraenkel set theory with choice aka 'the' foundation of mathematics		
Hilbert-Bernays's Grundlagen der Mathematik	medium	Z_2 (second-order arithmetic) ⋮ Π^1_2-CA_0 (comprehension for Π^1_2-formulas) Π^1_1-CA_0 (comprehension for Π^1_1-formulas) ATR_0 (arithmetical transfinite recursion) ACA_0 (arithmetical comprehension)
Russell-Weyl-Feferman predicative mathematics		
	weak	WKL_0 (weak König's lemma) RCA_0 (recursive comprehension) PRA (primitive recursive arithmetic) bounded arithmetic

Gödel hierarchy

strong

Zermelo-Fraenkel set theory with choice
aka 'the' foundation of mathematics

- ⋮
- large cardinals
- ⋮
- ZFC
- ZC (Zermelo set theory)
- simple type theory

medium

Hilbert-Bernays's Grundlagen der Mathematik

Russell-Weyl-Feferman predicative mathematics

- Z_2 (second-order arithmetic)
- ⋮
- Π^1_2-CA_0 (comprehension for Π^1_2-formulas)
- Π^1_1-CA_0 (comprehension for Π^1_1-formulas)
- ATR_0 (arithmetical transfinite recursion)
- ACA_0 (arithmetical comprehension)

weak

The 'Big Five' of Reverse Mathematics: Π^1_1-CA_0, ATR_0, ACA_0, WKL_0, RCA_0

- WKL_0 (weak König's lemma)
- RCA_0 (recursive comprehension)
- PRA (primitive recursive arithmetic)
- bounded arithmetic

Gödel hierarchy

	strong	⋮ large cardinals ⋮ ZFC
Zermelo-Fraenkel set theory with choice aka 'the' foundation of mathematics		ZC (Zermelo set theory) simple type theory
Hilbert-Bernays's Grundlagen der Mathematik	medium	Z_2 (second-order arithmetic) ⋮ Π^1_2-CA_0 (comprehension for Π^1_2-formulas)
Russell-Weyl-Feferman predicative mathematics		Π^1_1-CA_0 (comprehension for Π^1_1-formulas) ATR_0 (arithmetical transfinite recursion) ACA_0 (arithmetical comprehension)
The 'Big Five' of Reverse Mathematics		
Hilbert's finitary math	weak	WKL_0 (weak König's lemma) RCA_0 (recursive comprehension) PRA (primitive recursive arithmetic) bounded arithmetic

Gödel hierarchy

strong

- ⋮
- large cardinals
- ⋮
- ZFC
- ZC (Zermelo set theory)
- simple type theory

Zermelo-Fraenkel set theory with choice aka 'the' foundation of mathematics

medium

- Z_2 (second-order arithmetic)
- ⋮
- Π^1_2-CA_0 (comprehension for Π^1_2-formulas)
- Π^1_1-CA_0 (comprehension for Π^1_1-formulas)
- ATR_0 (arithmetical transfinite recursion)
- ACA_0 (arithmetical comprehension)

Hilbert-Bernays's Grundlagen der Mathematik

Russell-Weyl-Feferman predicative mathematics

The 'Big Five' of Reverse Mathematics

weak

- WKL_0 (weak König's lemma)
- RCA_0 (recursive comprehension)
- PRA (primitive recursive arithmetic)
- bounded arithmetic

Hilbert's finitary math

Received view: natural/important systems form linear Gödel hierarchy

Gödel hierarchy

	strong	⋮ large cardinals ⋮ ZFC
Zermelo-Fraenkel set theory with choice aka 'the' foundation of mathematics		ZC (Zermelo set theory) simple type theory
Hilbert-Bernays's Grundlagen der Mathematik	medium	Z_2 (second-order arithmetic) ⋮ Π^1_2-CA_0 (comprehension for Π^1_2-formulas)
Russell-Weyl-Feferman predicative mathematics		Π^1_1-CA_0 (comprehension for Π^1_1-formulas) ATR_0 (arithmetical transfinite recursion) ACA_0 (arithmetical comprehension)
The 'Big Five' of Reverse Mathematics		
Hilbert's finitary math	weak	WKL_0 (weak König's lemma) RCA_0 (recursive comprehension) PRA (primitive recursive arithmetic) bounded arithmetic

Received view: natural/important systems form linear Gödel hierarchy and 80/90% of ordinary mathematics is provable in ACA_0/Π^1_1-CA_0.

History of comprehension (and vice versa)

History of comprehension (and vice versa)

In Grundlagen der Mathematik, Hilbert and Bernays formalise (a lot of) mathematics in a logical system H.

History of comprehension (and vice versa)

In Grundlagen der Mathematik, Hilbert and Bernays formalise (a lot of) mathematics in a logical system H.

System H makes (essential) use of third-order parameters, but is 'more second-order' than previous systems (with Ackermann).

History of comprehension (and vice versa)

In Grundlagen der Mathematik, Hilbert and Bernays formalise (a lot of) mathematics in a logical system H.

System H makes (essential) use of third-order parameters, but is 'more second-order' than previous systems (with Ackermann).

H inspired second-order arithmetic Z_2 based on comprehension:

$$(\exists X \subset \mathbb{N})(\forall n \in \mathbb{N})(n \in X \leftrightarrow \varphi(n))$$

for any formula $\varphi(n)$ in L_2, language of Z_2.

History of comprehension (and vice versa)

In Grundlagen der Mathematik, Hilbert and Bernays formalise (a lot of) mathematics in a logical system H.

System H makes (essential) use of third-order parameters, but is 'more second-order' than previous systems (with Ackermann).

H inspired second-order arithmetic Z_2 based on comprehension:

$$(\exists X \subset \mathbb{N})(\forall n \in \mathbb{N})(n \in X \leftrightarrow \varphi(n))$$

for any formula $\varphi(n)$ in L_2, language of Z_2.

Indeed, the following is (explicitly) introduced in H:

$$(\exists n \in \mathbb{N})(f(n) = 0) \to f(\mu(f)) = 0 \quad \text{(Feferman's } \mu)$$

History of comprehension (and vice versa)

In Grundlagen der Mathematik, Hilbert and Bernays formalise (a lot of) mathematics in a logical system H.

System H makes (essential) use of third-order parameters, but is 'more second-order' than previous systems (with Ackermann).

H inspired second-order arithmetic Z_2 based on comprehension:

$$(\exists X \subset \mathbb{N})(\forall n \in \mathbb{N})(n \in X \leftrightarrow \varphi(n))$$

for any formula $\varphi(n)$ in L_2, language of Z_2.

Indeed, the following is (explicitly) introduced in H:

$$(\exists n \in \mathbb{N})(f(n) = 0) \rightarrow f(\mu(f)) = 0 \quad \text{(Feferman's } \mu\text{)}$$

yielding arithmetical comprehension as in ACA_0.

History of comprehension (and vice versa)

In Grundlagen der Mathematik, Hilbert and Bernays formalise (a lot of) mathematics in a logical system H.

System H makes (essential) use of third-order parameters, but is 'more second-order' than previous systems (with Ackermann).

H inspired second-order arithmetic Z_2 based on comprehension:

$$(\exists X \subset \mathbb{N})(\forall n \in \mathbb{N})(n \in X \leftrightarrow \varphi(n))$$

for any formula $\varphi(n)$ in L_2, language of Z_2.

Indeed, the following is (explicitly) introduced in H:

$$(\exists n \in \mathbb{N})(f(n) = 0) \to f(\mu(f)) = 0 \quad \text{(Feferman's } \mu\text{)}$$

yielding arithmetical comprehension as in ACA_0. Similarly:

ν-functional produces witness to $(\exists f : \mathbb{N} \to \mathbb{N})A(f)$, yielding Z_2.

Comprehension by any other name

Comprehension by any other name

Z_2 is based on comprehension as follows:

$$(\exists X \subset \mathbb{N})(\forall n \in \mathbb{N})(n \in X \leftrightarrow \varphi(n))$$

for any formula $\varphi(n)$ in L_2, language of Z_2. (Kreisel?)

Comprehension by any other name

Z_2 is based on comprehension as follows:

$$(\exists X \subset \mathbb{N})(\forall n \in \mathbb{N})(n \in X \leftrightarrow \varphi(n))$$

for any formula $\varphi(n)$ in L_2, language of Z_2. (Kreisel?)

Z_2^ω is based on comprehension as follows:

$$(\exists f : \mathbb{N} \to \mathbb{N})A(f) \leftrightarrow A(\nu_{k+1} g.A(g)) \tag{*}$$

for $A \in \Pi_k^1 \cap L_2$ and any k. (Feferman, Sieg, Suslin, Kohlenbach)

Comprehension by any other name

Z_2 is based on comprehension as follows:

$$(\exists X \subset \mathbb{N})(\forall n \in \mathbb{N})(n \in X \leftrightarrow \varphi(n))$$

for any formula $\varphi(n)$ in L_2, language of Z_2. (Kreisel?)

Z_2^ω is based on comprehension as follows:

$$(\exists f : \mathbb{N} \to \mathbb{N})A(f) \leftrightarrow A(\nu_{k+1} g . A(g)) \qquad (*)$$

for $A \in \Pi_k^1 \cap L_2$ and any k. (Feferman, Sieg, Suslin, Kohlenbach)

Z_2^Ω is based on comprehension as follows:

$$(\exists f : \mathbb{N} \to \mathbb{N})(Y(f) = 0) \leftrightarrow E(Y) = 0.$$

for any third-order $Y : \mathbb{N}^\mathbb{N} \to \mathbb{N}$.

Comprehension by any other name

Z_2 is based on comprehension as follows:

$$(\exists X \subset \mathbb{N})(\forall n \in \mathbb{N})(n \in X \leftrightarrow \varphi(n))$$

for any formula $\varphi(n)$ in L_2, language of Z_2. (Kreisel?)

Z_2^ω is based on comprehension as follows:

$$(\exists f : \mathbb{N} \to \mathbb{N})A(f) \leftrightarrow A(\nu_{k+1} g.A(g)) \qquad (*)$$

for $A \in \Pi_k^1 \cap L_2$ and any k. (Feferman, Sieg, Suslin, Kohlenbach)

Z_2^Ω is based on comprehension as follows:

$$(\exists f : \mathbb{N} \to \mathbb{N})(Y(f) = 0) \leftrightarrow E(Y) = 0.$$

for any third-order $Y : \mathbb{N}^\mathbb{N} \to \mathbb{N}$. E is called Kleene's $\exists^3$.

Comprehension by any other name

Z_2 is based on comprehension as follows:

$$(\exists X \subset \mathbb{N})(\forall n \in \mathbb{N})(n \in X \leftrightarrow \varphi(n))$$

for any formula $\varphi(n)$ in L_2, language of Z_2. (Kreisel?)

Z_2^ω is based on comprehension as follows:

$$(\exists f : \mathbb{N} \to \mathbb{N})A(f) \leftrightarrow A(\nu_{k+1}g.A(g)) \qquad (*)$$

for $A \in \Pi_k^1 \cap L_2$ and any k. (Feferman, Sieg, Suslin, Kohlenbach)

Z_2^Ω is based on comprehension as follows:

$$(\exists f : \mathbb{N} \to \mathbb{N})(Y(f) = 0) \leftrightarrow E(Y) = 0.$$

for any third-order $Y : \mathbb{N}^\mathbb{N} \to \mathbb{N}$. E is called Kleene's $\exists^3$.

Connection: $Z_2 \equiv_{L_2} Z_2^\omega \equiv_{L_2} Z_2^\Omega$.

Comprehension by any other name

Z_2 is based on comprehension as follows:

$$(\exists X \subset \mathbb{N})(\forall n \in \mathbb{N})(n \in X \leftrightarrow \varphi(n))$$

for any formula $\varphi(n)$ in L_2, language of Z_2. (Kreisel?)

Z_2^ω is based on comprehension as follows:

$$(\exists f : \mathbb{N} \to \mathbb{N})A(f) \leftrightarrow A(\nu_{k+1} g.A(g)) \qquad (*)$$

for $A \in \Pi_k^1 \cap L_2$ and any k. (Feferman, Sieg, Suslin, Kohlenbach)

Z_2^Ω is based on comprehension as follows:

$$(\exists f : \mathbb{N} \to \mathbb{N})(Y(f) = 0) \leftrightarrow E(Y) = 0.$$

for any third-order $Y : \mathbb{N}^\mathbb{N} \to \mathbb{N}$. E is called Kleene's $\exists^3$.

Connection: $Z_2 \equiv_{L_2} Z_2^\omega \equiv_{L_2} Z_2^\Omega$. Note 3rd vs 4th order!

Incomprehensible!

Recall that $\mathsf{Z}_2 \equiv_{L_2} \mathsf{Z}_2^{\omega} \equiv_{L_2} \mathsf{Z}_2^{\Omega}$.

Incomprehensible!

Recall that $Z_2 \equiv_{L_2} Z_2^\omega \equiv_{L_2} Z_2^\Omega$. The following *third-order* theorems are provable in Z_2^Ω, but not in Z_2^ω.

Incomprehensible!

Recall that $\mathsf{Z}_2 \equiv_{L_2} \mathsf{Z}_2^\omega \equiv_{L_2} \mathsf{Z}_2^\Omega$. The following *third-order* theorems are provable in Z_2^Ω, but not in Z_2^ω.

1. Arzelà's convergence theorem for Riemann integral (1885).

Incomprehensible!

Recall that $\mathsf{Z}_2 \equiv_{L_2} \mathsf{Z}_2^{\omega} \equiv_{L_2} \mathsf{Z}_2^{\Omega}$. The following *third-order* theorems are provable in Z_2^{Ω}, but not in Z_2^{ω}.

1. Arzelà's convergence theorem for Riemann integral (1885).
2. A countably-compact metric space $([0,1], d)$ is separable.

Incomprehensible!

Recall that $\mathsf{Z}_2 \equiv_{L_2} \mathsf{Z}_2^{\omega} \equiv_{L_2} \mathsf{Z}_2^{\Omega}$. The following *third-order* theorems are provable in Z_2^{Ω}, but not in Z_2^{ω}.

1. Arzelà's convergence theorem for Riemann integral (1885).
2. A countably-compact metric space $([0, 1], d)$ is separable.
3. Baire category theorem (open sets as characteristic functions)

Incomprehensible!

Recall that $Z_2 \equiv_{L_2} Z_2^\omega \equiv_{L_2} Z_2^\Omega$. The following *third-order* theorems are provable in Z_2^Ω, but not in Z_2^ω.

1. Arzelà's convergence theorem for Riemann integral (1885).
2. A countably-compact metric space $([0,1], d)$ is separable.
3. Baire category theorem (open sets as characteristic functions)
4. There is a function $f : \mathbb{R} \to \mathbb{R}$ not in Baire class 2.

Incomprehensible!

Recall that $\mathsf{Z}_2 \equiv_{L_2} \mathsf{Z}_2^\omega \equiv_{L_2} \mathsf{Z}_2^\Omega$. The following *third-order* theorems are provable in Z_2^Ω, but not in Z_2^ω.

1. Arzelà's convergence theorem for Riemann integral (1885).
2. A countably-compact metric space $([0,1], d)$ is separable.
3. Baire category theorem (open sets as characteristic functions)
4. There is a function $f : \mathbb{R} \to \mathbb{R}$ not in Baire class 2.
5. Baire characterisation theorem for Baire class 1.

Incomprehensible!

Recall that $Z_2 \equiv_{L_2} Z_2^\omega \equiv_{L_2} Z_2^\Omega$. The following *third-order* theorems are provable in Z_2^Ω, but not in Z_2^ω.

1. Arzelà's convergence theorem for Riemann integral (1885).
2. A countably-compact metric space $([0,1], d)$ is separable.
3. Baire category theorem (open sets as characteristic functions)
4. There is a function $f : \mathbb{R} \to \mathbb{R}$ not in Baire class 2.
5. Baire characterisation theorem for Baire class 1.
6. Heine-Borel/Vitali/Lindelöf for uncountable coverings.

Incomprehensible!

Recall that $\mathsf{Z}_2 \equiv_{L_2} \mathsf{Z}_2^{\omega} \equiv_{L_2} \mathsf{Z}_2^{\Omega}$. The following *third-order* theorems are provable in Z_2^{Ω}, but not in Z_2^{ω}.

1. Arzelà's convergence theorem for Riemann integral (1885).
2. A countably-compact metric space $([0,1], d)$ is separable.
3. Baire category theorem (open sets as characteristic functions)
4. There is a function $f : \mathbb{R} \to \mathbb{R}$ not in Baire class 2.
5. Baire characterisation theorem for Baire class 1.
6. Heine-Borel/Vitali/Lindelöf for uncountable coverings.
7. Basic Lebesgue measure/integral and gauge integral.

Incomprehensible!

Recall that $\mathsf{Z}_2 \equiv_{L_2} \mathsf{Z}_2^\omega \equiv_{L_2} \mathsf{Z}_2^\Omega$. The following *third-order* theorems are provable in Z_2^Ω, but not in Z_2^ω.

1. Arzelà's convergence theorem for Riemann integral (1885).
2. A countably-compact metric space $([0,1], d)$ is separable.
3. Baire category theorem (open sets as characteristic functions)
4. There is a function $f : \mathbb{R} \to \mathbb{R}$ not in Baire class 2.
5. Baire characterisation theorem for Baire class 1.
6. Heine-Borel/Vitali/Lindelöf for uncountable coverings.
7. Basic Lebesgue measure/integral and gauge integral.
8. Unordered sums $\sum_{x\in\mathbb{R}} f(x)$ are countable (E.H. Moore)

Incomprehensible!

Recall that $Z_2 \equiv_{L_2} Z_2^\omega \equiv_{L_2} Z_2^\Omega$. The following *third-order* theorems are provable in Z_2^Ω, but not in Z_2^ω.

1. Arzelà's convergence theorem for Riemann integral (1885).
2. A countably-compact metric space $([0, 1], d)$ is separable.
3. Baire category theorem (open sets as characteristic functions)
4. There is a function $f : \mathbb{R} \to \mathbb{R}$ not in Baire class 2.
5. Baire characterisation theorem for Baire class 1.
6. Heine-Borel/Vitali/Lindelöf for uncountable coverings.
7. Basic Lebesgue measure/integral and gauge integral.
8. Unordered sums $\sum_{x\in\mathbb{R}} f(x)$ are countable (E.H. Moore)
9. Convergence theorems for nets indexed by $\mathbb{N}^\mathbb{N}$ (Moore-Smith).

Incomprehensible!

Recall that $Z_2 \equiv_{L_2} Z_2^\omega \equiv_{L_2} Z_2^\Omega$. The following *third-order* theorems are provable in Z_2^Ω, but not in Z_2^ω.

1. Arzelà's convergence theorem for Riemann integral (1885).
2. A countably-compact metric space $([0,1], d)$ is separable.
3. Baire category theorem (open sets as characteristic functions)
4. There is a function $f : \mathbb{R} \to \mathbb{R}$ not in Baire class 2.
5. Baire characterisation theorem for Baire class 1.
6. Heine-Borel/Vitali/Lindelöf for uncountable coverings.
7. Basic Lebesgue measure/integral and gauge integral.
8. Unordered sums $\sum_{x\in\mathbb{R}} f(x)$ are countable (E.H. Moore)
9. Convergence theorems for nets indexed by $\mathbb{N}^\mathbb{N}$ (Moore-Smith).
10. The uncountability of $\mathbb{R}$: there is no injection (or bijection) from $[0,1]$ to $\mathbb{N}$ (Cantor, 1874).

Incomprehensible!

Recall that $Z_2 \equiv_{L_2} Z_2^\omega \equiv_{L_2} Z_2^\Omega$. The following *third-order* theorems are provable in Z_2^Ω, but not in Z_2^ω.

1. Arzelà's convergence theorem for Riemann integral (1885).
2. A countably-compact metric space $([0,1], d)$ is separable.
3. Baire category theorem (open sets as characteristic functions)
4. There is a function $f : \mathbb{R} \to \mathbb{R}$ not in Baire class 2.
5. Baire characterisation theorem for Baire class 1.
6. Heine-Borel/Vitali/Lindelöf for uncountable coverings.
7. Basic Lebesgue measure/integral and gauge integral.
8. Unordered sums $\sum_{x\in\mathbb{R}} f(x)$ are countable (E.H. Moore)
9. Convergence theorems for nets indexed by $\mathbb{N}^\mathbb{N}$ (Moore-Smith).
10. The uncountability of $\mathbb{R}$: there is no injection (or bijection) from $[0,1]$ to $\mathbb{N}$ (Cantor, 1874).
11. Basic RM theorems with usual definition of countable set.

Uncountability of $\mathbb{R}$

Uncountability of $\mathbb{R}$

Cantor (1874): for any sequence of reals $(x_n)_{n\in\mathbb{N}}$, there is $y \in \mathbb{R}$ such that $x_n \neq y$ for all $n \in \mathbb{N}$.

Uncountability of $\mathbb{R}$

Cantor (1874): for any sequence of reals $(x_n)_{n\in\mathbb{N}}$, there is $y \in \mathbb{R}$ such that $x_n \neq y$ for all $n \in \mathbb{N}$.

To avoid the anti-platonist ire of Kronecker-Weierstrass, Cantor (1874) only mentions that $\mathbb{R}$ and $\mathbb{N}$ are 'therefore' not one-to-one.

Uncountability of $\mathbb{R}$

Cantor (1874): for any sequence of reals $(x_n)_{n\in\mathbb{N}}$, there is $y \in \mathbb{R}$ such that $x_n \neq y$ for all $n \in \mathbb{N}$.

To avoid the anti-platonist ire of Kronecker-Weierstrass, Cantor (1874) only mentions that $\mathbb{R}$ and $\mathbb{N}$ are 'therefore' not one-to-one.

How hard is it to prove the 'real' uncountability of $\mathbb{R}$ as follows?

Uncountability of $\mathbb{R}$

Cantor (1874): for any sequence of reals $(x_n)_{n\in\mathbb{N}}$, there is $y \in \mathbb{R}$ such that $x_n \neq y$ for all $n \in \mathbb{N}$.

To avoid the anti-platonist ire of Kronecker-Weierstrass, Cantor (1874) only mentions that $\mathbb{R}$ and $\mathbb{N}$ are 'therefore' not one-to-one.

How hard is it to prove the 'real' uncountability of $\mathbb{R}$ as follows?

Theorem (NIN, see Kunen)

For $Y : [0,1] \to \mathbb{N}$, there are $x, y \in [0,1]$ s.t. $x \neq y \wedge Y(x) = Y(y)$

Theorem (NBI, see Hrbacek-Jech)

For $Y : [0,1] \to \mathbb{N}$, there are distinct $x, y \in [0,1]$ such that $Y(x) = Y(y)$ OR there is $n \in \mathbb{N}$ with $(\forall x \in [0,1])(Y(x) \neq n)$.

Uncountability of $\mathbb{R}$

Cantor (1874): for any sequence of reals $(x_n)_{n\in\mathbb{N}}$, there is $y \in \mathbb{R}$ such that $x_n \neq y$ for all $n \in \mathbb{N}$.

To avoid the anti-platonist ire of Kronecker-Weierstrass, Cantor (1874) only mentions that $\mathbb{R}$ and $\mathbb{N}$ are 'therefore' not one-to-one.

How hard is it to prove the 'real' uncountability of $\mathbb{R}$ as follows?

Theorem (NIN, see Kunen)

For $Y : [0,1] \to \mathbb{N}$, there are $x, y \in [0,1]$ s.t. $x \neq y \wedge Y(x) = Y(y)$

Theorem (NBI, see Hrbacek-Jech)

For $Y : [0,1] \to \mathbb{N}$, there are distinct $x, y \in [0,1]$ such that $Y(x) = Y(y)$ OR there is $n \in \mathbb{N}$ with $(\forall x \in [0,1])(Y(x) \neq n)$.

These are provable in Z_2^Ω but not in Z_2^ω (and the weakest such).

Two nice observations about the uncountability of $\mathbb{R}$

Firstly, $Z_2^\omega + \neg\mathsf{NBI}$ proves Z_2 and is consistent.

Two nice observations about the uncountability of $\mathbb{R}$

Firstly, $Z_2^\omega + \neg NBI$ proves Z_2 and is consistent. By $\neg NBI$, there is a bijection $Y : [0,1] \to \mathbb{N}$, i.e. there is a 'first' real x such that $Y(x) = 0$, a 'second' real y such that $Y(y) = 1$, et cetera.

Two nice observations about the uncountability of $\mathbb{R}$

Firstly, $Z_2^\omega + \neg\text{NBI}$ proves Z_2 and is consistent. By $\neg$NBI, there is a bijection $Y : [0,1] \to \mathbb{N}$, i.e. there is a 'first' real x such that $Y(x) = 0$, a 'second' real y such that $Y(y) = 1$, et cetera.

Hence, $\mathbb{R}$ is a potential infinity (following Stillwell) BUT one can develop second-order RM 'as usual'. Extraordinary math?

Two nice observations about the uncountability of $\mathbb{R}$

Firstly, $Z_2^\omega + \neg\text{NBI}$ proves Z_2 and is consistent. By $\neg$NBI, there is a bijection $Y : [0,1] \to \mathbb{N}$, i.e. there is a 'first' real x such that $Y(x) = 0$, a 'second' real y such that $Y(y) = 1$, et cetera.

Hence, $\mathbb{R}$ is a potential infinity (following Stillwell) BUT one can develop second-order RM 'as usual'. Extraordinary math?

History repeats itself: Borel and others objected against AC although their earlier work made (essential) us of AC.

Two nice observations about the uncountability of $\mathbb{R}$

Firstly, $Z_2^\omega + \neg\text{NBI}$ proves Z_2 and is consistent. By $\neg$NBI, there is a bijection $Y : [0,1] \to \mathbb{N}$, i.e. there is a 'first' real x such that $Y(x) = 0$, a 'second' real y such that $Y(y) = 1$, et cetera.

Hence, $\mathbb{R}$ is a potential infinity (following Stillwell) BUT one can develop second-order RM 'as usual'. Extraordinary math?

History repeats itself: Borel and others objected against AC although their earlier work made (essential) us of AC. Weierstrass rejected the idea that there can be different 'sizes' of infinity (like $\mathbb{N}$ and $\mathbb{R}$) although his earlier theorems imply NIN and NBI.

Two nice observations about the uncountability of $\mathbb{R}$

Firstly, $Z_2^\omega + \neg$NBI proves Z_2 and is consistent. By $\neg$NBI, there is a bijection $Y : [0,1] \to \mathbb{N}$, i.e. there is a 'first' real x such that $Y(x) = 0$, a 'second' real y such that $Y(y) = 1$, et cetera.

Hence, $\mathbb{R}$ is a potential infinity (following Stillwell) BUT one can develop second-order RM 'as usual'. Extraordinary math?

History repeats itself: Borel and others objected against AC although their earlier work made (essential) us of AC. Weierstrass rejected the idea that there can be different 'sizes' of infinity (like $\mathbb{N}$ and $\mathbb{R}$) although his earlier theorems imply NIN and NBI.

In contrast to the modern era, Weierstrass changed his mind in light of Cantor's work...

Countable sets versus sets that are countable

The word ‘countable’ appears hundreds of times in Simpson’s ‘bible of RM’;

Countable sets versus sets that are countable

The word ‘countable’ appears hundreds of times in Simpson’s ‘bible of RM’; here, ‘countable’ means ‘given by a sequence’.

Countable sets versus sets that are countable

The word ‘countable’ appears hundreds of times in Simpson’s ‘bible of RM’; here, ‘countable’ means ‘given by a sequence’.

What happens if we use the real (third-order) definition of ‘countable’?

Countable sets versus sets that are countable

The word 'countable' appears hundreds of times in Simpson's 'bible of RM'; here, 'countable' means 'given by a sequence'.

What happens if we use the real (third-order) definition of 'countable'? The following theorems are then provable in Z_2^Ω and not provable in Z_2^ω.

Countable sets versus sets that are countable

The word 'countable' appears hundreds of times in Simpson's 'bible of RM'; here, 'countable' means 'given by a sequence'.

What happens if we use the real (third-order) definition of 'countable'? The following theorems are then provable in Z_2^Ω and not provable in Z_2^ω.

(a) A countable subset of $[0, 1]$ has a supremum (Bolzano-Weierstrass).

Countable sets versus sets that are countable

The word 'countable' appears hundreds of times in Simpson's 'bible of RM'; here, 'countable' means 'given by a sequence'.

What happens if we use the real (third-order) definition of 'countable'? The following theorems are then provable in Z_2^Ω and not provable in Z_2^ω.

(a) A countable subset of $[0, 1]$ has a supremum (Bolzano-Weierstrass).

(b) A countable collection of basic open intervals covering $[0, 1]$, has a finite sub-cover. (Heine-Borel)

Countable sets versus sets that are countable

The word 'countable' appears hundreds of times in Simpson's 'bible of RM'; here, 'countable' means 'given by a sequence'.

What happens if we use the real (third-order) definition of 'countable'? The following theorems are then provable in Z_2^Ω and not provable in Z_2^ω.

(a) A countable subset of $[0, 1]$ has a supremum (Bolzano-Weierstrass).

(b) A countable collection of basic open intervals covering $[0, 1]$, has a finite sub-cover. (Heine-Borel)

(c) Vitali's covering theorem for countable collections.

Countable sets versus sets that are countable

The word 'countable' appears hundreds of times in Simpson's 'bible of RM'; here, 'countable' means 'given by a sequence'.

What happens if we use the real (third-order) definition of 'countable'? The following theorems are then provable in Z_2^{Ω} and not provable in Z_2^{ω}.

(a) A countable subset of $[0, 1]$ has a supremum (Bolzano-Weierstrass).

(b) A countable collection of basic open intervals covering $[0, 1]$, has a finite sub-cover. (Heine-Borel)

(c) Vitali's covering theorem for countable collections.

(d) A countable set in $\mathbb{R}$ has finite measure.

Countable sets versus sets that are countable

The word 'countable' appears hundreds of times in Simpson's 'bible of RM'; here, 'countable' means 'given by a sequence'.

What happens if we use the real (third-order) definition of 'countable'? The following theorems are then provable in Z_2^Ω and not provable in Z_2^ω.

(a) A countable subset of $[0, 1]$ has a supremum (Bolzano-Weierstrass).

(b) A countable collection of basic open intervals covering $[0, 1]$, has a finite sub-cover. (Heine-Borel)

(c) Vitali's covering theorem for countable collections.

(d) A countable set in $\mathbb{R}$ has finite measure.

(e) And probably everything else in RM mentioning 'countable'...

Countable sets versus sets that are countable

The word 'countable' appears hundreds of times in Simpson's 'bible of RM'; here, 'countable' means 'given by a sequence'.

What happens if we use the real (third-order) definition of 'countable'? The following theorems are then provable in Z_2^Ω and not provable in Z_2^ω.

(a) A countable subset of $[0,1]$ has a supremum (Bolzano-Weierstrass).
(b) A countable collection of basic open intervals covering $[0,1]$, has a finite sub-cover. (Heine-Borel)
(c) Vitali's covering theorem for countable collections.
(d) A countable set in $\mathbb{R}$ has finite measure.
(e) And probably everything else in RM mentioning 'countable'...

Explosion: Π_2^1-CA_0 follows from item (a) plus Π_1^1-CA_0^ω.

Countable sets versus sets that are countable

The word 'countable' appears hundreds of times in Simpson's 'bible of RM'; here, 'countable' means 'given by a sequence'.

What happens if we use the real (third-order) definition of 'countable'? The following theorems are then provable in Z_2^{Ω} and not provable in Z_2^{ω}.

(a) A countable subset of $[0,1]$ has a supremum (Bolzano-Weierstrass).

(b) A countable collection of basic open intervals covering $[0,1]$, has a finite sub-cover. (Heine-Borel)

(c) Vitali's covering theorem for countable collections.

(d) A countable set in $\mathbb{R}$ has finite measure.

(e) And probably everything else in RM mentioning 'countable'...

Explosion: $\Pi_2^1\text{-CA}_0$ follows from item (a) plus $\Pi_1^1\text{-CA}_0^{\omega}$.

Warning: same for 'countable' combinatorics and the RM zoo!

Problem, cause, and solution

Problem, cause, and solution

PROBLEM: hundreds of intuitively weak third-order theorems are classified as rather strong qua third-order comprehension, i.e. not provable in Z_2^ω and provable in Z_2^Ω, for $Z_2 \equiv_{L_2} Z_2^\omega \equiv_{L_2} Z_2^\Omega$.

Problem, cause, and solution

PROBLEM: hundreds of intuitively weak third-order theorems are classified as rather strong qua third-order comprehension, i.e. not provable in Z_2^ω and provable in Z_2^Ω, for $Z_2 \equiv_{L_2} Z_2^\omega \equiv_{L_2} Z_2^\Omega$.

CAUSE: comprehension functionals (like $\mu, \nu_n, \exists^3$) are discontinuous (or: normal).

Problem, cause, and solution

PROBLEM: hundreds of intuitively weak third-order theorems are classified as rather strong qua third-order comprehension, i.e. not provable in Z_2^ω and provable in Z_2^Ω, for $Z_2 \equiv_{L_2} Z_2^\omega \equiv_{L_2} Z_2^\Omega$.

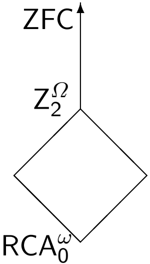
CAUSE: comprehension functionals (like $\mu, \nu_n, \exists^3$) are discontinuous (or: normal). The other theorems (uncountability of $\mathbb{R}$, Heine-Borel,...) are non-normal, i.e. consistent with Brouwer's continuity theorem that 'all functions are continuous'.

Problem, cause, and solution

PROBLEM: hundreds of intuitively weak third-order theorems are classified as rather strong qua third-order comprehension, i.e. not provable in Z_2^ω and provable in Z_2^Ω, for $Z_2 \equiv_{L_2} Z_2^\omega \equiv_{L_2} Z_2^\Omega$.

CAUSE: comprehension functionals (like $\mu, \nu_n, \exists^3$) are discontinuous (or: normal). The other theorems (uncountability of $\mathbb{R}$, Heine-Borel,...) are non-normal, i.e. consistent with Brouwer's continuity theorem that 'all functions are continuous'.

SOLUTION: split the hierarchy below Z_2^Ω in normal and non-normal part.



Normal part with hierarchy $\Pi_k^1\text{-CA}_0^\omega$ and discontinuous functionals ν_k. (Kohlenbach)

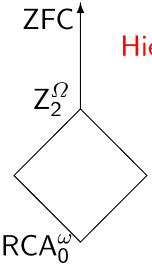
Non-normal part consistent with Brouwer's continuity theorem
(Heine-Borel, uncountability of $\mathbb{R}$, ...)

Problem, cause, and solution

PROBLEM: hundreds of intuitively weak third-order theorems are classified as rather strong qua third-order comprehension, i.e. not provable in Z_2^ω and provable in Z_2^Ω, for $Z_2 \equiv_{L_2} Z_2^\omega \equiv_{L_2} Z_2^\Omega$.

CAUSE: comprehension functionals (like $\mu, \nu_n, \exists^3$) are discontinuous (or: normal). The other theorems (uncountability of $\mathbb{R}$, Heine-Borel,…) are non-normal, i.e. consistent with Brouwer's continuity theorem that 'all functions are continuous'.

SOLUTION: split the hierarchy below Z_2^Ω in normal and non-normal part.



Hierarchy for non-normal part?

Normal part with hierarchy Π_k^1-CA_0^ω and discontinuous functionals ν_k. (Kohlenbach)

Non-normal part consistent with Brouwer's continuity theorem
(Heine-Borel, uncountability of $\mathbb{R}$, …)

Brouwer and continuity to the rescue

L.E.J. Brouwer is (in)famous for his *intuitionism*.

Brouwer and continuity to the rescue

L.E.J. Brouwer is (in)famous for his *intuitionism*.

Intuitionistic mathematics is formalised using non-classical continuity axioms that have a (non-classical) weak counterpart.

Brouwer and continuity to the rescue

L.E.J. Brouwer is (in)famous for his *intuitionism*.

Intuitionistic mathematics is formalised using non-classical continuity axioms that have a (non-classical) weak counterpart.

The 'weak' counterpart yields the usual axiom via the classically valid Neighbourhood Function Principle (NFP).

Brouwer and continuity to the rescue

L.E.J. Brouwer is (in)famous for his *intuitionism*.

Intuitionistic mathematics is formalised using non-classical continuity axioms that have a (non-classical) weak counterpart.

The 'weak' counterpart yields the usual axiom via the classically valid Neighbourhood Function Principle (NFP).

Definition (NFP, 1970, Kreisel-Troelstra)

For any formula A, we have

$$(\forall f \in \mathbb{N}^{\mathbb{N}})(\exists n \in \mathbb{N})A(\overline{f}n) \rightarrow (\exists \gamma \in K_0)(\forall f \in \mathbb{N}^{\mathbb{N}})A(\overline{f}\gamma(f)),$$

where '$\gamma \in K_0$' essentially means that γ is an RM-code/associate.

Note that $\overline{f}n$ is the finite sequence $\langle f(0), f(1), \ldots, f(n-1)\rangle$.

Brouwer and continuity to the rescue

L.E.J. Brouwer is (in)famous for his *intuitionism*.

Intuitionistic mathematics is formalised using non-classical continuity axioms that have a (non-classical) weak counterpart.

The 'weak' counterpart yields the usual axiom via the classically valid Neighbourhood Function Principle (NFP).

Definition (NFP, 1970, Kreisel-Troelstra)

For any formula A, we have

$$(\forall f \in \mathbb{N}^{\mathbb{N}})(\exists n \in \mathbb{N})A(\overline{f}n) \rightarrow (\exists \gamma \in K_0)(\forall f \in \mathbb{N}^{\mathbb{N}})A(\overline{f}\gamma(f)),$$

where '$\gamma \in K_0$' essentially means that γ is an RM-code/associate.

Note that $\overline{f}n$ is the finite sequence $\langle f(0), f(1), \ldots, f(n-1)\rangle$.

NFP expresses that there are (many) continuous choice functions.

Brouwer and continuity to the rescue

Definition (NFP, 1970, Kreisel-Troelstra)

For any formula A, we have

$$(\forall f \in \mathbb{N}^{\mathbb{N}})(\exists n \in \mathbb{N})A(\overline{f}n) \to (\exists \gamma \in K_0)(\forall f \in \mathbb{N}^{\mathbb{N}})A(\overline{f}\gamma(f)),$$

where '$\gamma \in K_0$' essentially means that γ is an RM-code/associate.

NFP has great properties (in contrast to comprehension):

Brouwer and continuity to the rescue

Definition (NFP, 1970, Kreisel-Troelstra)

For any formula A, we have

$$(\forall f \in \mathbb{N}^{\mathbb{N}})(\exists n \in \mathbb{N})A(\overline{f}n) \rightarrow (\exists \gamma \in K_0)(\forall f \in \mathbb{N}^{\mathbb{N}})A(\overline{f}\gamma(f)),$$

where '$\gamma \in K_0$' essentially means that γ is an RM-code/associate.

NFP has great properties (in contrast to comprehension):

1) Many non-normal theorems (Heine-Borel, Lindeloef, monotone convergence theorem for nets, ...) are equivalent to natural fragments of NFP.

Brouwer and continuity to the rescue

Definition (NFP, 1970, Kreisel-Troelstra)

For any formula A, we have

$$(\forall f \in \mathbb{N}^{\mathbb{N}})(\exists n \in \mathbb{N})A(\overline{f}n) \rightarrow (\exists \gamma \in K_0)(\forall f \in \mathbb{N}^{\mathbb{N}})A(\overline{f}\gamma(f)),$$

where '$\gamma \in K_0$' essentially means that γ is an RM-code/associate.

NFP has great properties (in contrast to comprehension):

1) Many non-normal theorems (Heine-Borel, Lindeloef, monotone convergence theorem for nets, ...) are equivalent to natural fragments of NFP.

2) The equivalences from 1) map to the Big Five equivalences, under the canonical embedding of HOA in SOA.

Brouwer and continuity to the rescue

Definition (NFP, 1970, Kreisel-Troelstra)

For any formula A, we have

$$(\forall f \in \mathbb{N}^{\mathbb{N}})(\exists n \in \mathbb{N})A(\overline{f}n) \rightarrow (\exists \gamma \in K_0)(\forall f \in \mathbb{N}^{\mathbb{N}})A(\overline{f}\gamma(f)),$$

where '$\gamma \in K_0$' essentially means that γ is an RM-code/associate.

NFP has great properties (in contrast to comprehension):

1) Many non-normal theorems (Heine-Borel, Lindeloef, monotone convergence theorem for nets, ...) are equivalent to natural fragments of NFP.

2) The equivalences from 1) map to the Big Five equivalences, under the canonical embedding of HOA in SOA.

The second item reminds one of Plato's allegory of the cave.

Plato and his -ism

Plato and his -ism

Plato is well-known in (foundations of) mathematics for his eponymous philosophy platonism, i.e.

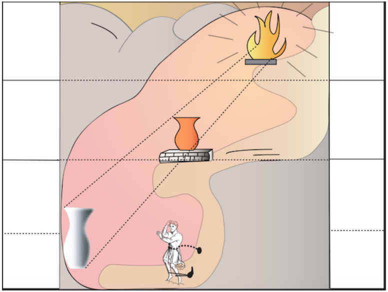
the theory that mathematical objects are objective, timeless entities, independent of the physical world and the symbols that represent them.

Plato and his -ism

Plato is well-known in (foundations of) mathematics for his eponymous philosophy platonism, i.e.

the theory that mathematical objects are objective, timeless entities, independent of the physical world and the symbols that represent them.

Plato's allegory of the cave provides a powerful visual:

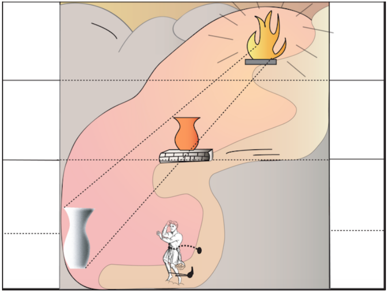


Plato and his -ism

Plato is well-known in (foundations of) mathematics for his eponymous philosophy platonism, i.e.

the theory that mathematical objects are objective, timeless entities, independent of the physical world and the symbols that represent them.

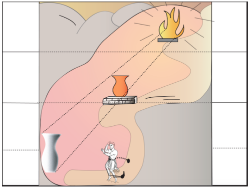
Plato's allegory of the cave provides a powerful visual:



We can only know reflections/shadows/... of ideal objects.

Plato and his -ism

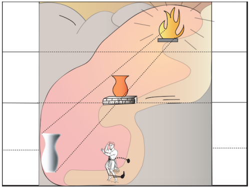
Plato's allegory of the cave provides a powerful visual:



We can only know reflections/shadows/... of ideal objects.

Plato and his -ism

Plato's allegory of the cave provides a powerful visual:

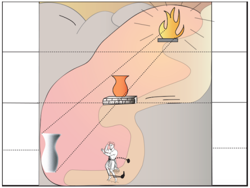


We can only know reflections/shadows/... of ideal objects.

What are the current foundations of mathematics reflections of?

Plato and his -ism

Plato's allegory of the cave provides a powerful visual:



We can only know reflections/shadows/... of ideal objects.

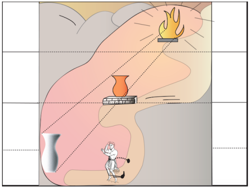
What are the current foundations of mathematics reflections of?

Fragments of NFP
and equivalences

Big Five
and equivalences

Plato and his -ism

Plato's allegory of the cave provides a powerful visual:



We can only know reflections/shadows/... of ideal objects.

What are the current foundations of mathematics reflections of?

Fragments of NFP
and equivalences

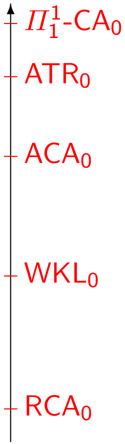
↙ ECF

Big Five
and equivalences

ECF is canonical embedding of HOA into SOA (Kleene-Kreisel).

The Big Five as a reflection

The Big Five as a reflection



The Big Five as a reflection

Π^1_1-CA_0

ATR_0

ACA_0

WKL_0

RCA_0 proves Δ^0_1-comprehension

The Big Five as a reflection

Π_1^1-CA_0

ATR_0

ACA_0

WKL_0 $\leftrightarrow$ Dini's theorem.
$\leftrightarrow$ countabe Heine-Borel compactness
$\leftrightarrow$ Riemann int. thms

RCA_0 proves Δ_1^0-comprehension

The Big Five as a reflection

Π_1^1-CA_0

ATR_0

ACA_0
$\leftrightarrow$ range of $f : \mathbb{N} \to \mathbb{N}$ exists
$\leftrightarrow$ Monotone conv. thm
$\leftrightarrow$ Ascoli-Arzela
$\leftrightarrow$ thms about closed sets (as countable unions)

WKL_0
$\leftrightarrow$ Dini's theorem.
$\leftrightarrow$ countabe Heine-Borel compactness
$\leftrightarrow$ Riemann int. thms

RCA_0 proves Δ_1^0-comprehension

The Big Five as a reflection

Π^1_1-CA_0

ATR_0 $\leftrightarrow$ perfect set theorem

ACA_0
$\leftrightarrow$ range of $f : \mathbb{N} \to \mathbb{N}$ exists
$\leftrightarrow$ Monotone conv. thm
$\leftrightarrow$ Ascoli-Arzela
$\leftrightarrow$ thms about closed sets (as countable unions)

WKL_0
$\leftrightarrow$ Dini's theorem.
$\leftrightarrow$ countabe Heine-Borel compactness
$\leftrightarrow$ Riemann int. thms

RCA_0 proves Δ^0_1-comprehension

The Big Five as a reflection

- Π^1_1-CA$_0$ $\leftrightarrow$ Cantor-Bendixson thm
- ATR$_0$ $\leftrightarrow$ perfect set theorem
- ACA$_0$
  - $\leftrightarrow$ range of $f : \mathbb{N} \rightarrow \mathbb{N}$ exists
  - $\leftrightarrow$ Monotone conv. thm
  - $\leftrightarrow$ Ascoli-Arzela
  - $\leftrightarrow$ thms about closed sets (as countable unions)
- WKL$_0$
  - $\leftrightarrow$ Dini's theorem.
  - $\leftrightarrow$ countabe Heine-Borel compactness
  - $\leftrightarrow$ Riemann int. thms
- RCA$_0$ proves Δ^0_1-comprehension

SECOND-ORDER arithmetic

The Big Five as a reflection

Π_1^1-CA$_0$ $\leftrightarrow$ Cantor-Bendixson thm

ATR$_0$ $\leftrightarrow$ perfect set theorem

$\leftrightarrow$ range of $f : \mathbb{N} \to \mathbb{N}$ exists
ACA$_0$ $\leftrightarrow$ Monotone conv. thm
$\leftrightarrow$ Ascoli-Arzela
$\leftrightarrow$ thms about closed sets (as countable unions)

WKL$_0$ $\leftrightarrow$ Dini's theorem.
$\leftrightarrow$ countabe Heine-Borel compactness
$\leftrightarrow$ Riemann int. thms

RCA$_0$ proves Δ_1^0-comprehension

SECOND-ORDER arithmetic

BOOT$_2$

Σ-TR

BOOT

WKL^1

RCA$_0^\omega$

HIGHER-ORDER arithmetic

The Big Five as a reflection

SECOND-ORDER arithmetic

- Π^1_1-CA$_0$ $\leftrightarrow$ Cantor-Bendixson thm
- ATR$_0$ $\leftrightarrow$ perfect set theorem
- ACA$_0$
  - $\leftrightarrow$ range of $f : \mathbb{N} \to \mathbb{N}$ exists
  - $\leftrightarrow$ Monotone conv. thm
  - $\leftrightarrow$ Ascoli-Arzela
  - $\leftrightarrow$ thms about closed sets (as countable unions)
- WKL$_0$
  - $\leftrightarrow$ Dini's theorem.
  - $\leftrightarrow$ countabe Heine-Borel compactness
  - $\leftrightarrow$ Riemann int. thms
- RCA$_0$ proves Δ^0_1-comprehension

HIGHER-ORDER arithmetic

- BOOT$_2$
- Σ-TR
- BOOT
- WKL^1
- RCA$_0^\omega$ plus a fragment of countable choice

The Big Five as a reflection

Π_1^1-CA_0 $\leftrightarrow$ Cantor-Bendixson thm

ATR_0 $\leftrightarrow$ perfect set theorem

ACA_0 $\leftrightarrow$ range of $f : \mathbb{N} \to \mathbb{N}$ exists
$\leftrightarrow$ Monotone conv. thm
$\leftrightarrow$ Ascoli-Arzela
$\leftrightarrow$ thms about closed sets (as countable unions)

WKL_0 $\leftrightarrow$ Dini's theorem.
$\leftrightarrow$ countabe Heine-Borel compactness
$\leftrightarrow$ Riemann int. thms

RCA_0 proves Δ_1^0-comprehension

SECOND-ORDER arithmetic

BOOT_2

Σ-TR

BOOT

WKL^1 $\leftrightarrow$ Dini's theorem for nets.
$\leftrightarrow$ uncountabe Heine-Borel compactness: HBU
$\leftrightarrow$ gauge integral thms

RCA_0^ω plus a fragment of countable choice

HIGHER-ORDER arithmetic

The Big Five as a reflection

Π_1^1-CA_0 $\leftrightarrow$ Cantor-Bendixson thm

ATR_0 $\leftrightarrow$ perfect set theorem

ACA_0
$\leftrightarrow$ range of $f : \mathbb{N} \to \mathbb{N}$ exists
$\leftrightarrow$ Monotone conv. thm
$\leftrightarrow$ Ascoli-Arzela
$\leftrightarrow$ thms about closed sets (as countable unions)

WKL_0
$\leftrightarrow$ Dini's theorem.
$\leftrightarrow$ countabe Heine-Borel compactness
$\leftrightarrow$ Riemann int. thms

RCA_0 proves Δ_1^0-comprehension

SECOND-ORDER arithmetic

BOOT_2

Σ-TR

BOOT
$\leftrightarrow$ range of $Y : \mathbb{N}^{\mathbb{N}} \to \mathbb{N}$ exist
$\leftrightarrow$ Mon. conv. thm for nets
$\leftrightarrow$ Ascoli-Arzela for nets
$\leftrightarrow$ thms about closed sets (as uncountable unions)

WKL^1
$\leftrightarrow$ Dini's theorem for nets.
$\leftrightarrow$ uncountabe Heine-Borel compactness: HBU
$\leftrightarrow$ gauge integral thms

RCA_0^ω plus a fragment of countable choice

HIGHER-ORDER arithmetic

The Big Five as a reflection

SECOND-ORDER arithmetic

- Π^1_1-CA_0 $\leftrightarrow$ Cantor-Bendixson thm
- ATR_0 $\leftrightarrow$ perfect set theorem
- ACA_0
  - $\leftrightarrow$ range of $f : \mathbb{N} \to \mathbb{N}$ exists
  - $\leftrightarrow$ Monotone conv. thm
  - $\leftrightarrow$ Ascoli-Arzela
  - $\leftrightarrow$ thms about closed sets (as countable unions)
- WKL_0
  - $\leftrightarrow$ Dini's theorem.
  - $\leftrightarrow$ countabe Heine-Borel compactness
  - $\leftrightarrow$ Riemann int. thms
- RCA_0 proves Δ^0_1-comprehension

HIGHER-ORDER arithmetic

- BOOT_2 $\leftrightarrow$ Cantor-Bendixson thm (uncountable unions)
- Σ-TR
- BOOT
  - $\leftrightarrow$ range of $Y : \mathbb{N}^{\mathbb{N}} \to \mathbb{N}$ exist
  - $\leftrightarrow$ Mon. conv. thm for nets
  - $\leftrightarrow$ Ascoli-Arzela for nets
  - $\leftrightarrow$ thms about closed sets (as uncountable unions)
- WKL^1
  - $\leftrightarrow$ Dini's theorem for nets.
  - $\leftrightarrow$ uncountabe Heine-Borel compactness: HBU
  - $\leftrightarrow$ gauge integral thms
- RCA^ω_0 plus a fragment of countable choice

The Big Five as a reflection

SECOND-ORDER arithmetic

- Π_1^1-CA_0 ↔ Cantor-Bendixson thm
- ATR_0 ↔ perfect set theorem
- ACA_0
  - ↔ range of $f : \mathbb{N} \to \mathbb{N}$ exists
  - ↔ Monotone conv. thm
  - ↔ Ascoli-Arzela
  - ↔ thms about closed sets (as countable unions)
- WKL_0
  - ↔ Dini's theorem.
  - ↔ countabe Heine-Borel compactness
  - ↔ Riemann int. thms
- RCA_0 proves Δ_1^0-comprehension

HIGHER-ORDER arithmetic

- BOOT_2 ↔ Cantor-Bendixson thm (uncountable unions)
- Σ-TR ↔ perfect set thm (idem)
- BOOT
  - ↔ range of $Y : \mathbb{N}^{\mathbb{N}} \to \mathbb{N}$ exist
  - ↔ Mon. conv. thm for nets
  - ↔ Ascoli-Arzela for nets
  - ↔ thms about closed sets (as uncountable unions)
- WKL^1
  - ↔ Dini's theorem for nets.
  - ↔ uncountabe Heine-Borel compactness: HBU
  - ↔ gauge integral thms
- RCA_0^ω plus a fragment of countable choice

The Big Five as a reflection

ECF replaces uncountable objects by countable representations/RM-codes

SECOND-ORDER arithmetic

- Π^1_1-CA$_0$ $\leftrightarrow$ Cantor-Bendixson thm
- ATR$_0$ $\leftrightarrow$ perfect set theorem
- ACA$_0$ $\leftrightarrow$ range of $f : \mathbb{N} \to \mathbb{N}$ exists
  $\leftrightarrow$ Monotone conv. thm
  $\leftrightarrow$ Ascoli-Arzela
  $\leftrightarrow$ thms about closed sets (as countable unions)
- WKL$_0$ $\leftrightarrow$ Dini's theorem.
  $\leftrightarrow$ countabe Heine-Borel compactness
  $\leftrightarrow$ Riemann int. thms
- RCA$_0$ proves Δ^0_1-comprehension

HIGHER-ORDER arithmetic

- BOOT$_2$ $\leftrightarrow$ Cantor-Bendixson thm (uncountable unions)
- Σ-TR $\leftrightarrow$ perfect set thm (idem)
- BOOT $\leftrightarrow$ range of $Y : \mathbb{N}^{\mathbb{N}} \to \mathbb{N}$ exist
  $\leftrightarrow$ Mon. conv. thm for nets
  $\leftrightarrow$ Ascoli-Arzela for nets
  $\leftrightarrow$ thms about closed sets (as uncountable unions)
- WKL^1 $\leftrightarrow$ Dini's theorem for nets.
  $\leftrightarrow$ uncountabe Heine-Borel compactness: HBU
  $\leftrightarrow$ gauge integral thms
- RCA$_0^\omega$ plus a fragment of countable choice

ECF $\leftarrow$

The Big Five as a reflection

ECF replaces uncountable objects by countable representations/RM-codes

ECF converts right-hand side to left-hand side, including equivalences!

SECOND-ORDER arithmetic		ECF ←	HIGHER-ORDER arithmetic	
Π^1_1-CA$_0$	↔ Cantor-Bendixson thm		BOOT$_2$	↔ Cantor-Bendixson thm (uncountable unions)
ATR$_0$	↔ perfect set theorem		Σ-TR	↔ perfect set thm (idem)
ACA$_0$	↔ range of $f : \mathbb{N} \to \mathbb{N}$ exists ↔ Monotone conv. thm ↔ Ascoli-Arzela ↔ thms about closed sets (as countable unions)	ECF ←	BOOT	↔ range of $Y : \mathbb{N}^{\mathbb{N}} \to \mathbb{N}$ exist ↔ Mon. conv. thm for nets ↔ Ascoli-Arzela for nets ↔ thms about closed sets (as uncountable unions)
WKL$_0$	↔ Dini's theorem. ↔ countabe Heine-Borel compactness ↔ Riemann int. thms		WKL^1	↔ Dini's theorem for nets. ↔ uncountabe Heine-Borel compactness: HBU ↔ gauge integral thms
RCA$_0$	proves Δ^0_1-comprehension	ECF ←	RCA$^\omega_0$	plus a fragment of countable choice

SECOND-ORDER arithmetic ECF ← HIGHER-ORDER arithmetic

Foundations/philosophy of mathematics

One can (and people probably will) argue forever which '-ism' is the true foundations/philosophy of mathematics.

Foundations/philosophy of mathematics

One can (and people probably will) argue forever which '-ism' is the true foundations/philosophy of mathematics.

One could also take a hint from the exact sciences (to which math technically belongs) and try to find evidence in support of one's viewpoint.

Foundations/philosophy of mathematics

One can (and people probably will) argue forever which '-ism' is the true foundations/philosophy of mathematics.

One could also take a hint from the exact sciences (to which math technically belongs) and try to find evidence in support of one's viewpoint.

I present the previous picture as evidence supporting Platonism.

Conclusion

Conclusion

Coding in L_2 is not bad *per se*: it works for continuous functions, but is a bad idea for discontinuous functions from the pov of RM.

Conclusion

Coding in L_2 is not bad *per se*: it works for continuous functions, but is a bad idea for discontinuous functions from the pov of RM. This is witnessed by basic theorems, like Arzela's convergence thm for the Riemann integral.

Conclusion

Coding in L_2 is not bad *per se*: it works for continuous functions, but is a bad idea for discontinuous functions from the pov of RM. This is witnessed by basic theorems, like Arzela's convergence thm for the Riemann integral.

To properly study discontinuous functions, one adopts Kohlenbach's higher-order RM. This 'normal' scale however classifies 'intuitively weak' theorems as 'rather strong', including the uncountability of $\mathbb{R}$.

Conclusion

Coding in L_2 is not bad *per se*: it works for continuous functions, but is a bad idea for discontinuous functions from the pov of RM. This is witnessed by basic theorems, like Arzela's convergence thm for the Riemann integral.

To properly study discontinuous functions, one adopts Kohlenbach's higher-order RM. This 'normal' scale however classifies 'intuitively weak' theorems as 'rather strong', including the uncountability of $\mathbb{R}$.

To solve this problem, one adopts the complimentary non-normal scale based on classically valid continuity axioms (NFP) from Brouwer's intuitionistic mathematics.

In the spirit of Plato's cave, the Big Five of RM are a reflection of the non-normal scale under Kleene-Kreisel's ECF.

Final Thoughts

Final Thoughts

The revolution is not an apple that falls when it is ripe. You have to make it fall. (AN & CG)

Final Thoughts

The revolution is not an apple that falls when it is ripe. You have to make it fall. (AN & CG)

The safest general characterisation of the European philosophical tradition is that it consists of a series of footnotes to Plato. (A.N. Whitehead)

The Lebesgue integral in mathematics is perhaps best compared to the imperial system in the USA. (anon)

Final Thoughts

The revolution is not an apple that falls when it is ripe. You have to make it fall. (AN & CG)

The safest general characterisation of the European philosophical tradition is that it consists of a series of footnotes to Plato. (A.N. Whitehead)

The Lebesgue integral in mathematics is perhaps best compared to the imperial system in the USA. (anon)

We thank DFG, TU Darmstadt, John Templeton Foundation, and Alexander Von Humboldt Foundation for their generous support!

Final Thoughts

The revolution is not an apple that falls when it is ripe. You have to make it fall. (AN & CG)

The safest general characterisation of the European philosophical tradition is that it consists of a series of footnotes to Plato. (A.N. Whitehead)

The Lebesgue integral in mathematics is perhaps best compared to the imperial system in the USA. (anon)

We thank DFG, TU Darmstadt, John Templeton Foundation, and Alexander Von Humboldt Foundation for their generous support!

Thank you for your attention!

Final Thoughts

The revolution is not an apple that falls when it is ripe. You have to make it fall. (AN & CG)

The safest general characterisation of the European philosophical tradition is that it consists of a series of footnotes to Plato. (A.N. Whitehead)

The Lebesgue integral in mathematics is perhaps best compared to the imperial system in the USA. (anon)

We thank DFG, TU Darmstadt, John Templeton Foundation, and Alexander Von Humboldt Foundation for their generous support!

Thank you for your attention!

Any (content) questions?

Raphael's Annotated School of Athens